A RECURSION FORMULA FOR THE CORRELATION FUNCTIONS OF AN INHOMOGENEOUS XXX MODEL

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Dedicated to Ludwig Faddeev on the occasion of his seventieth birthday

ABSTRACT. A new recursion formula is presented for the correlation functions of the integrable spin 1/2 XXX chain with inhomogeneity. It relates the correlators involving n consecutive lattice sites to those with $n-1$ and $n-2$ sites. In a series of papers by V. Korepin and two of the present authors, it was discovered that the correlators have a certain specific structure as functions of the inhomogeneity parameters. Our formula allows for a direct proof of this structure, as well as an exact description of the rational functions which has been left undetermined in the previous works.

CONTENTS

1. Introduction

Consider the XXX anti-ferromagnet given by the Hamiltonian

$$H_{XXX} = \frac{1}{2} \sum_j \left(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \sigma_j^z \sigma_{j+1}^z \right).$$

This model was solved in the famous paper by Bethe [3] already in 1931, using what is now called the coordinate Bethe Ansatz. Nevertheless it took some time before the physical content of the model in the thermodynamic limit was completely clarified. The spectrum of excitations was correctly described for the first time in the paper by Faddeev and Takhtajan [10]; it was shown that the spectrum contains magnons of spin $1/2$. These authors used the algebraic Bethe Ansatz formulated by Faddeev, Sklyanin and Takhtajan (see the review [9]) on the basis of R-matrices and the Yang-Baxter equation. The origin of these new techniques goes back to the works of Baxter [1].

Let us recall on our example the role of R-matrices in solvable models. The XXX model is related to the rational R-matrix which acts on $\mathbb{C}^2 \otimes \mathbb{C}^2$ (see (2.7) for the explicit formula). We use the usual notation $R_{1,2}(\lambda)$ where $1, 2$ label the corresponding spaces and λ is the spectral parameter. The relation of the R-matrix to the XXX Hamiltonian is as follows. Consider the transfer matrix

$$t_N(\lambda) = \text{tr}\left(R_{\alpha,-N}(\lambda) R_{\alpha,-N+1}(\lambda) \cdots R_{\alpha,N}(\lambda) \right),$$

where the trace is taken with respect to the auxiliary space labelled by α. The transfer matrices commute for different values of spectral parameters, giving rise to a commuting family of operators. If we expand $t_N(\lambda)$ in powers of λ,

$$\log t_N(0)^{-1} t_N(\lambda) = \sum_{k=1}^{\infty} I_k \lambda^k,$$

then I_1 coincides with the XXX Hamiltonian for the periodic chain of length $2N+1$. All other I_k are integrals of motion commuting with I_1.

We have repeated these well-known facts in order to make clear the following remark due to Baxter. Consider the inhomogeneous chain whose integrals of motion are generated by the transfer matrix

$$\text{tr}\left(R_{a,-N}(\lambda) \cdots R_{a,0}(\lambda) R_{a,1}(\lambda - \lambda_1) \cdots R_{a,n}(\lambda - \lambda_n) R_{a,n+1}(\lambda) \cdots R_{a,N}(\lambda) \right),$$

where $\lambda_1, \cdots \lambda_n$ are arbitrary parameters. The model is still exactly solvable, although the interaction is not local any more. This generalization will be very important for us.

After the calculation of the spectrum, the next important issue is that of the correlation functions. Following [12], we consider general correlators of the form

$$\langle \text{vac}|(E_{\epsilon_1,\bar{\epsilon}_1})_1 \cdots (E_{\epsilon_n,\bar{\epsilon}_n})_n|\text{vac}\rangle.$$

These are the averages over the ground state $|\text{vac}\rangle$ of products of elementary operators $(E_{\epsilon_j,\bar{\epsilon}_j})_j$ $(\epsilon_j, \bar{\epsilon}_j = \pm 1)$ at site j. For $n = 2$, they can be calculated easily from the vacuum energy. The first non-trivial result is due to Takahashi [21], who evaluated the correlators for $n = 3$ in terms of $\zeta(3)$, where ζ is the Riemann ζ-function.

Results for general n were brought forth 15 years later by Jimbo, Miki, Miwa and Nakayashiki [13], in the framework of representation theory of quantum affine algebras. Their results and further developments are presented in the book [12]. It was shown in the context of a more general XXZ model that the correlators in both homogeneous and inhomogeneous cases are given in terms of multiple integrals in which the number of integrations is equal to the distance n on the lattice. Actually, the inhomogeneous case plays a very important role. The correlators in this case depend on the parameters λ_j since the vacuum does. As functions of λ_j, the correlators satisfy the quantum Knizhnik-Zamolodchikov equation (qKZ) [11] with level -4. Here an unexpected similarity became apparent between the correlators in lattice models and form factors in integrable relativistic models calculated by Smirnov [19]. The latter also satisfy the qKZ equation but with level 0. The symmetry algebra of the XXX model is the Yangian, and the qKZ equation in this situation has been discussed in [20].

One remark is in order here. The algebraic methods of [12] work nicely in the presence of a gap in the spectrum. In the gapless case (such as the XXX model under consideration), the formulae for the correlators were obtained by "analytical continuation". However, later the same formulae were derived rigorously by Kitanine, Maillet and Terras [16] on the basis of the algebraic Bethe Ansatz.

Actually, it is not very simple to obtain the result of Takahashi from the formulas in [13]. For $n = 3$, we have three-fold integrals, and the result must be surprisingly simple. This observation was the starting point of the works by Boos and Korepin [4]. They were able to calculate further the cases $n = 4$ and $n = 5$ (for some correlators). In all cases the multiple integrals mysteriously disappeared, and the results were expressed in terms of products of ζ-functions at odd positive integers with rational coefficients. This mystery had to be explained.

The explanation was presented in the series of papers by Boos, Korepin and Smirnov [5, 6, 7]. The idea is again the use of inhomogeneous model and the qKZ equation. In the paper [5] was put forward a conjecture which states that all the correlators are expressed schematically as follows.

$$(1.1) \qquad \langle \mathrm{vac} | (E_{\epsilon_1, \bar{\epsilon}_1})_1 \cdots (E_{\epsilon_n, \bar{\epsilon}_n})_n | \mathrm{vac} \rangle = \sum \prod \omega(\lambda_i - \lambda_j)\, f(\lambda_1, \cdots, \lambda_n).$$

Here $\omega(\lambda)$ is a certain transcendental function (a linear combination of logarithmic derivatives of the Γ-function), the $f(\lambda_1, \cdots, \lambda_n)$ are rational functions, and the sum is taken over partitions (for the precise formula, see (2.4) and Theorem 3.2, 3.3). In the homogeneous limit, odd integer values of the ζ-function appear as the coefficients in the Taylor series of $\omega(\lambda)$.

It was explained that the real reason for the formula (1.1) to hold is the existence of a formula for solutions to the qKZ equation with level -4 different than those given in [12]. This follows from a duality between the level 0 and level -4 cases. In the case of level 0, all solutions in the $\mathfrak{sl}_2$-invariant subspace are known. To get solutions for the level -4 case, one has to invert the matrix of solutions for the level 0 case. This can be done efficiently due to the relation to the symplectic group.

Detailed explanations of this point will bring us too far from the subject of the present paper.

Let us consider (1.1) as an Ansatz. The question is to calculate the rational functions $f(\lambda_1, \cdots, \lambda_n)$. It turns out that this problem is much more complicated than it appears at a first glance. Despite the efforts made in the papers [5, 6, 7], only partial answers were gained for small n: All the correlation functions for the homogeneous case of the XXX model until $n = 4$ were calculated by K.Sakai, M.Shiroishi, Y.Nishiyama and M.Takahashi [18] while the result for those functions in the XXZ case was done in the papers [14], [22], [15]. The whole set of correlation functions for $n = 5$ both in homogeneous and inhomogeneous cases has been recently found in [8].

The main results of the present paper are

  (i) Calculation of the functions $f(\lambda_1, \cdots, \lambda_n)$,
  (ii) Proof of the Ansatz (1.1).

Surprisingly enough, the result is expressed in terms of transfer matrices over an auxiliary space of 'fractional dimension'. This brings us very close to the theory of the Baxter Q-operators developed by Bazhanov, Lukyanov and Zamolodchikov [2]. We hope to discuss this issue in future publications.

The text is organized as follows.

We begin in 2.1 by formulating the problem of computing the correlation functions. We review the Ansatz of [5, 6, 7]. In 2.2 we prepare basic materials from the algebraic Bethe Ansatz. In 2.3 we summarize the properties of the solution to the qKZ equation relevant to the correlators.

We introduce in 3.1 the trace over a space of 'fractional dimension'. Using this notion, we define in 3.2 the rational function $X^{[i,j]}$ which enter the recursion formula as the coefficients. The recursion is stated in 3.3. Simple examples are given in 3.4 for the correlation functions obtained from the recursion.

In 4.1, we give several properties of $X^{[i,j]}$. Using these properties and the recursion, we prove the Ansatz in 4.2. The proof of recursion is started in 4.3 by calculating the residues of the correlation functions. The proof is completed in 4.4 by giving an asymptotic estimate.

In Appendix A we discuss the relations between the different gauges of the Hamiltonian used here and in the literature. In Appendix B we give a proof of the analytic and asymptotic properties of the correlators used in the text.

2. Correlation functions and reduced qKZ equation

2.1. **Formulation of the problem.** In this subsection, we formulate the problem we are going to address.

Consider the XXX model with the Hamiltonian

$$(2.1) \qquad H_{XXX} = \frac{1}{2} \sum_j \left(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \sigma_j^z \sigma_{j+1}^z \right).$$

Its integrability is due to the Yang-Baxter relation. The problem is to calculate the correlation functions

$$\langle \text{vac}|(E_{\epsilon_1,\bar{\epsilon}_1})_1 \cdots (E_{\epsilon_n,\bar{\epsilon}_n})_n|\text{vac}\rangle.$$

These are the averages over the ground state $|\text{vac}\rangle$ of products of elementary operators

$$E_{\epsilon_j,\bar{\epsilon}_j} = \left(\delta_{\epsilon_j\alpha}\delta_{\bar{\epsilon}_j\beta}\right)_{\alpha,\beta=\pm} \qquad (\epsilon_j,\bar{\epsilon}_j = \pm)$$

acting on the site j. More precisely, we consider the correlation functions of an inhomogeneous model, in which each site j carries an independent spectral parameter λ_j. An exact integral formula for these quantities has been known [12, 16].

It is convenient to pass to the quantity

(2.2)
$$h_n(\lambda_1,\cdots,\lambda_n)^{-\epsilon_1,\cdots,-\epsilon_n,\bar{\epsilon}_n,\cdots,\bar{\epsilon}_1} = (-1)^{[n/2]}\prod_{j=1}^{n}(-\bar{\epsilon}_j)\langle\text{vac}|(E_{\epsilon_1,\bar{\epsilon}_1})_1 \cdots (E_{\epsilon_n,\bar{\epsilon}_n})_n|\text{vac}\rangle.$$

In Appendix A, we give a connection of this formula to a similar formula for the XXZ model given in [12].

Denoting by v_+, v_- the standard basis of

$$V = \mathbb{C}^2 = \mathbb{C}v_+ \oplus \mathbb{C}v_-,$$

we regard

$$h_n(\lambda_1,\cdots,\lambda_n)$$
$$= \sum_{\epsilon_1,\cdots,\epsilon_n,\bar{\epsilon}_1,\cdots,\bar{\epsilon}_n} h_n(\lambda_1,\cdots,\lambda_n)^{\epsilon_1,\cdots,\epsilon_n,\bar{\epsilon}_n,\cdots,\bar{\epsilon}_1}v_{\epsilon_1}\otimes\cdots v_{\epsilon_n}\otimes v_{\bar{\epsilon}_n}\otimes\cdots\otimes v_{\bar{\epsilon}_1}$$

as an element of $V^{\otimes 2n}$. This function is obtained from a solution of the qKZ equation with level -4

$$g_{2n}(\lambda_1,\cdots,\lambda_{2n}) \in V^{\otimes 2n}$$

by specializing the arguments as (see e.g. [12])

(2.3) $$h_n(\lambda_1,\cdots,\lambda_n) = g_{2n}(\lambda_1,\cdots,\lambda_n,\lambda_n+1,\cdots,\lambda_1+1).$$

In [5, 6], it was found that the functions h_n have the following structure.

(2.4) $$h_n(\lambda_1,\cdots,\lambda_n) = \sum_{m=0}^{[n/2]}\sum_{I,J}\prod_{p=1}^{m}\omega(\lambda_{i_p}-\lambda_{j_p})\,f_{n,I,J}(\lambda_1,\cdots,\lambda_n).$$

Here $I = (i_1,\ldots,i_m)$, $J = (j_1,\ldots,j_m)$ and the sum is taken over all sequences I, J such that $I \cap J = \emptyset$, $1 \le i_p < j_p \le n$ $(1 \le p \le m)$ and $i_1 < \cdots < i_m$.

A characteristic feature of the formula (2.4) is that the transcendental functions enter only through the single function [1]

(2.5) $$\omega(\lambda) = (\lambda^2-1)\frac{d}{d\lambda}\log\rho(\lambda) + \frac{1}{2} = \sum_{k=1}^{\infty}(-1)^k\frac{2k(\lambda^2-1)}{\lambda^2-k^2} + \frac{1}{2},$$

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[1]The function $\omega(\lambda)$ is related to $G(\lambda)$ in [5] by $\omega(\lambda) = G(i\lambda) + 1/2$.

where

$$(2.6) \qquad \rho(\lambda) = -\frac{\Gamma\left(\frac{\lambda}{2}\right)\Gamma\left(-\frac{\lambda}{2} + \frac{1}{2}\right)}{\Gamma\left(-\frac{\lambda}{2}\right)\Gamma\left(\frac{\lambda}{2} + \frac{1}{2}\right)}.$$

The remaining factors $f_{n,I,J}(\lambda_1, \cdots, \lambda_n)$ are rational functions of $\lambda_1, \cdots, \lambda_n$ with only simple poles along the diagonal $\lambda_i = \lambda_j$. Subsequently it was explained [6, 7] that this structure of h_n originates from a duality between solutions of the qKZ equations with level 0 and level -4. On the basis of this relation, (2.4) was derived under certain assumptions. In these works, the rational functions $f_{n,I,J}$ have been left undetermined. Our goal in this paper is to derive a recursion formula for h_n, which enables us to describe $f_{n,I,J}$ and to give a direct proof of (2.4).

2.2. L operators and fusion of R matrices.
In this subsection, we introduce our notation concerning R matrices and L operators, and give several formulae for the fusion of R matrices.

The basic R matrix relevant to the XXX model is

$$(2.7) \qquad R(\lambda) = \rho(\lambda)\frac{r(\lambda)}{\lambda + 1},$$

where $\rho(\lambda)$ is given by (2.6),

$$r(\lambda) = \lambda + P,$$

and $P \in \mathrm{End}\left((\mathbb{C}^2)^{\otimes 2}\right)$ is the permutation operator, $P(u \otimes v) = v \otimes u$.

We consider also R matrices associated with higher dimensional representations of $\mathfrak{sl}_2$. Let us fix the convention as follows. We denote by

$$\pi^{(k)} \, : \, U(\mathfrak{sl}_2) \longrightarrow \mathrm{End}(V^{(k)}), \quad V^{(k)} \simeq \mathbb{C}^{k+1}$$

the $(k+1)$-dimensional irreducible representation of $\mathfrak{sl}_2$. We choose a basis $\{v_j^{(k)}\}_{j=0}^k$ of $V^{(k)}$ on which the standard generators E, F, H act as

$$Ev_j^{(k)} = (k - j + 1)v_{j-1}^{(k)}, \quad Fv_j^{(k)} = (j + 1)v_{j+1}^{(k)}, \quad Hv_j^{(k)} = (k - 2j)v_j^{(k)}$$

with $v_{-1}^{(k)} = v_{k+1}^{(k)} = 0$. For $k = 1$, we also write $V = V^{(1)}$ and $v_+ = v_0^{(1)}$, $v_- = v_1^{(1)}$. We shall use the singlet vectors in $V^{(k)} \otimes V^{(k)}$ $(k = 1, 2)$ normalized as

$$(2.8) \qquad s^{(1)} = v_+ \otimes v_- - v_- \otimes v_+,$$

$$(2.9) \qquad s^{(2)} = v_0^{(2)} \otimes v_2^{(2)} - \frac{1}{2}v_1^{(2)} \otimes v_1^{(2)} + v_2^{(2)} \otimes v_0^{(2)}.$$

Let $\{S_a\}_{a=1}^3$, $\{S^a\}_{a=1}^3$ be a dual basis of $\mathfrak{sl}_2$ with respect to the invariant bilinear form $(x|y)$ normalized as $(H|H) = 2$. It is well known that the element

$$(2.10) \qquad L^{(1)}(\lambda) = \lambda + \frac{1}{2} + \sum_{a=1}^3 S_a \otimes \pi^{(1)}(S^a) \quad \in U(\mathfrak{sl}_2) \otimes \mathrm{End}(V^{(1)})$$

is a solution of the Yang-Baxter relation

$$R_{1,2}(\lambda_1 - \lambda_2)L_1^{(1)}(\lambda_1)L_2^{(1)}(\lambda_2) = L_2^{(1)}(\lambda_2)L_1^{(1)}(\lambda_1)R_{1,2}(\lambda_1 - \lambda_2).$$

The suffix stands for the tensor components on which the operators act non-trivially. We also use a suffix of the form $(\alpha_1, \ldots, \alpha_k)$ to denote the symmetric part $V^{(k)} \subset V_{\alpha_1} \otimes \cdots \otimes V_{\alpha_k}$, where V_{α_i} are copies of V.

Identifying $V^{(2)}$ with the subspace $V^{(2)}_{(1,2)} = \mathcal{P}^+_{1,2}(V_1 \otimes V_2)$ where $\mathcal{P}^\pm = (1/2)(1 \pm P)$, we define the fused L-operator

$$(2.11) \quad L^{(2)}_{(1,2)}(\lambda) = L^{(1)}_1\Big(\lambda - \frac{1}{2}\Big) L^{(1)}_2\Big(\lambda + \frac{1}{2}\Big) \mathcal{P}^+_{1,2} \quad \in U(\mathfrak{sl}_2) \otimes \operatorname{End}(V^{(2)}).$$

Explicitly it is given by

$$(2.12) \qquad L^{(2)}(\lambda) = \lambda(\lambda+1) - \frac{1}{2} C \otimes \operatorname{id}_{V^{(2)}}$$

$$+ (\lambda+1)\sum_{a=1}^{3} S_a \otimes \pi^{(2)}(S^a) + \frac{1}{2}\sum_{a,b=1}^{3} S_a S_b \otimes \pi^{(2)}(S^a S^b).$$

In the right hand side,

$$(2.13) \qquad\qquad C = \sum_{a=1}^{3} S_a S^a$$

denotes the Casimir operator.

We shall make use of the crossing symmetry

$$(2.14) \qquad\qquad L^{(k)}_\alpha(\lambda) s^{(k)}_{\alpha,\beta} = L^{(k)}_\beta(-\lambda-1) s^{(k)}_{\beta,\alpha}$$

and the quantum determinant relation

$$(2.15) \qquad\qquad \mathcal{P}^-_{1,2} L^{(1)}_1(\lambda-1) L^{(1)}_2(\lambda) = \Big(\lambda^2 - \frac{1}{4} - \frac{1}{2}C\Big)\mathcal{P}^-_{1,2}.$$

Taking the images of (2.10), (2.11) in $V^{(k)}$, we obtain (numerical) R matrices [2]

$$(2.16) \qquad\qquad r^{(k,l)}(\lambda) = (\pi^{(k)} \otimes \operatorname{id})L^{(l)}(\lambda) \quad \in \operatorname{End}\big(V^{(k)} \otimes V^{(l)}\big).$$

In the following, we abbreviate $L^{(1)}(\lambda)$ to $L(\lambda)$.

We prepare several formulas about the fusion of R matrices. We have

$$(2.17) \quad r_{1,\alpha}\Big(\lambda - \frac{k-1}{2}\Big) r_{2,\alpha}\Big(\lambda - \frac{k-3}{2}\Big) \cdots r_{k,\alpha}\Big(\lambda + \frac{k-1}{2}\Big) \mathcal{P}^+_{1,\cdots,k} = c_k(\lambda) r^{(k,1)}_{(1,\cdots,k),\alpha}(\lambda)$$

where

$$(2.18) \qquad\qquad c_k(\lambda) = \prod_{j=1}^{k-1}\Big(\lambda - \frac{k-1}{2} + j\Big).$$

We have also for all $k \geq 1$

$$(2.19) \qquad r^{(k,1)}_{(\alpha_1,\cdots,\alpha_k),\alpha}\Big(\lambda - \frac{1}{2}\Big) r^{(k,1)}_{(\alpha_1,\cdots,\alpha_k),\beta}\Big(\lambda + \frac{1}{2}\Big) \mathcal{P}^+_{\alpha\beta} = r^{(k,2)}_{(\alpha_1,\cdots,\alpha_k),(\alpha\beta)}(\lambda)$$

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[2]We have $r^{(1,2)}_{3,(12)}(\lambda) = (\lambda + \frac{1}{2}) r^{(2,1)}_{(12),3}(\lambda)$.

and the relation

$$(2.20) \qquad (\lambda + \frac{k_1 - k_2}{2})(\lambda + 1 + \frac{k_1 - k_2}{2})r^{(k_1+k_2,2)}_{(1,\dots,k_1+k_2),(\alpha\beta)}(\lambda)$$

$$= r^{(k_1,2)}_{(1,\dots,k_1),(\alpha\beta)}(\lambda - \frac{k_2}{2})r^{(k_2,2)}_{(k_1+1,\dots,k_1+k_2),(\alpha\beta)}(\lambda + \frac{k_1}{2}).$$

2.3. **Properties of $h_n(\lambda_1,\dots,\lambda_n)$.** The function h_n is given by a specialization (2.3) of the solution g_{2n} of the qKZ equation with level -4. In this subsection, we summarize the properties of h_n implied by the ones of g_{2n}.

In what follows, we deal with various vectors in the tensor product $V^{\otimes 2n}$ and those obtained by permuting the tensor components. In order to simplify the presentation, we adopt the following convention. Consider the tensor product

$$(2.21) \qquad W = V_1 \otimes \cdots \otimes V_n \otimes V_{\bar{n}} \otimes \cdots \otimes V_{\bar{1}}$$

of $2n$ copies of V labeled by $1,\dots,n,\bar{n},\dots,\bar{1}$. For a vector

$$(2.22) \qquad f = \sum f^{\epsilon_1,\dots,\epsilon_n,\bar{\epsilon}_n,\dots,\bar{\epsilon}_1}v_{\epsilon_1} \otimes \dots \otimes v_{\epsilon_n} \otimes v_{\bar{\epsilon}_n} \otimes \dots \otimes v_{\bar{\epsilon}_1}$$

in $V^{\otimes 2n}$ we denote by $f_{1,\dots,n,\bar{n},\dots,\bar{1}}$ the same vector (2.22) in (2.21). Vectors obtained by permuting tensor components will be indicated by permuting suffixes from the 'standard position'. For example, if

$$f = \sum f^{\epsilon_1,\epsilon_2,\bar{\epsilon}_2,\bar{\epsilon}_1}v_{\epsilon_1} \otimes v_{\epsilon_2} \otimes v_{\bar{\epsilon}_2} \otimes v_{\bar{\epsilon}_1} \in V^{\otimes 4},$$

then $f_{2,\bar{2},1,\bar{1}} \in V_1 \otimes V_2 \otimes V_{\bar{2}} \otimes V_{\bar{1}}$ is given by

$$\begin{aligned} f_{2,\bar{2},1,\bar{1}} &= P_{1,\bar{2}}P_{1,2}f_{1,2,\bar{2},\bar{1}} \\ &= \sum f^{\epsilon_2,\bar{\epsilon}_2,\epsilon_1,\bar{\epsilon}_1}v_{\epsilon_1} \otimes v_{\epsilon_2} \otimes v_{\bar{\epsilon}_2} \otimes v_{\bar{\epsilon}_1}. \end{aligned}$$

By $f_{2,\bar{2},1,\bar{1}}$ we do not mean a vector in $V_2 \otimes V_{\bar{2}} \otimes V_1 \otimes V_{\bar{1}}$. Notice that in this notation $s^{(1)}_{\bar{1},1} = -s^{(1)}_{1,\bar{1}} \in V_1 \otimes V_{\bar{1}}$. Sometimes, we need to construct a vector in $V_1 \otimes V_2 \otimes V_{\bar{2}} \otimes V_{\bar{1}}$ from one in $V_1 \otimes V_{\bar{1}}$ and one in $V_2 \otimes V_{\bar{2}}$. Suppose that $f,g \in V \otimes V$. Then, we have $f_{1,\bar{1}} \in V_1 \otimes V_{\bar{1}}$, and $g_{2,\bar{2}} \in V_2 \otimes V_{\bar{2}}$. We write $f_{1,\bar{1}}g_{2,\bar{2}}$ for the vector

$$\sum f^{\epsilon_1,\bar{\epsilon}_1}g^{\epsilon_2,\bar{\epsilon}_2}v_{\epsilon_1} \otimes v_{\epsilon_2} \otimes v_{\bar{\epsilon}_2} \otimes v_{\bar{\epsilon}_1} \in V_1 \otimes V_2 \otimes V_{\bar{2}} \otimes V_{\bar{1}}.$$

In this convention, ordering of the tensor product is irrelevant. There is no preference in writing $f_{1,\bar{1}}g_{2,\bar{2}}$ or $g_{2,\bar{2}}f_{1,\bar{1}}$ to represent the above vector.

We have three more remarks. First, we use "auxiliary" spaces in addition to the "quantum" spaces $V_1, V_{\bar{1}}, \dots, V_n, V_{\bar{n}}$. We use $\alpha, \beta, \alpha_1, \alpha_2$, etc., to label these spaces. Second, we use the index with parenthesis like $(\alpha_1,\dots,\alpha_k)$ to label the completely symmetric subspace $V^{(k)} \subset V^{\otimes k}$. This was already mentioned in Section 2.2. If f is a vector in this subspace, we denote by $f_{(\alpha_1,\dots,\alpha_k)}$ the corresponding vector in $V_{\alpha_1} \otimes \cdots \otimes V_{\alpha_k}$. Lastly, we use similar convention for matrices as well.

The function g_{2n} has the following properties [12]:

$(2.23) g_{2n}(\lambda_1, \cdots, \lambda_{2n})$ is invariant under the action of $\mathfrak{sl}_2$,

$(2.24) g_{2n}(\cdots, \lambda_{j+1}, \lambda_j, \cdots)_{\cdots, j+1, j, \cdots} = R_{j,j+1}(\lambda_{j,j+1}) g_{2n}(\cdots, \lambda_j, \lambda_{j+1}, \cdots)_{\cdots, j, j+1, \cdots}$,

$(2.25) g_{2n}(\lambda_1, \cdots, \lambda_{2n-1}, \lambda_{2n} + 2)_{1, \cdots, 2n} = (-1)^n g_{2n}(\lambda_{2n}, \lambda_1, \cdots, \lambda_{2n-1})_{2n, 1, \cdots, 2n-1}$,

$(2.26) g_{2n}(\lambda_1, \cdots, \lambda_{2n-2}, \lambda, \lambda - 1)_{1, \cdots, 2n} = g_{2n-2}(\lambda_1, \cdots, \lambda_{2n-2})_{1, \cdots, 2n-2} \, s_{2n-1, 2n}^{(1)}$.

Note that $R(-1) = -1 + P = -2\mathcal{P}^-$ and $s_{21}^{(1)} = -s_{12}^{(1)}$. From this, (2.24) and (2.26), we have

$$\mathcal{P}_{2n-1, 2n}^- g_{2n}(\lambda_1, \cdots, \lambda_{2n-2}, \lambda, \lambda + 1)_{1, \cdots, 2n} = -\frac{1}{2} g_{2n-2}(\lambda_1, \cdots, \lambda_{2n-2})_{1, \cdots, 2n-2} \, s_{2n-1, 2n}^{(1)}.$$

Here and after we set $\lambda_{i,j} = \lambda_i - \lambda_j$. For $\alpha = 1$ or $\bar{1}$, set

$$(2.27) \qquad A_\alpha(\lambda_1, \cdots, \lambda_n)$$
$$= (-1)^n R_{\alpha, \bar{2}}(\lambda_{1,2} - 1) \cdots R_{\alpha, \bar{n}}(\lambda_{1,n} - 1) R_{\alpha, n}(\lambda_{1,n}) \cdots R_{\alpha, 2}(\lambda_{1,2}).$$

It is easy to prove the following proposition from the formulae above:

Proposition 2.1. *The function $h_n(\lambda_1, \cdots, \lambda_n)$ satisfies the following properties:*

$$(2.28) \qquad\qquad h_n(\lambda_1, \cdots, \lambda_n) \text{ is invariant under the action of } \mathfrak{sl}_2 ,$$

$$(2.29) \qquad h_n(\cdots, \lambda_{j+1}, \lambda_j, \cdots)_{\cdots, j+1, j, \cdots, \bar{j}, \overline{j+1}, \cdots}$$
$$= R_{j,j+1}(\lambda_{j,j+1}) R_{\overline{j+1}, \bar{j}}(\lambda_{j+1, j}) h_n(\cdots, \lambda_j, \lambda_{j+1}, \cdots)_{\cdots, j, j+1, \cdots, \overline{j+1}, \bar{j}, \cdots},$$

$$(2.30) \qquad\qquad h_n(\lambda_1 - 1, \lambda_2, \cdots, \lambda_n)_{1, \cdots, n, \bar{n}, \cdots, \bar{1}}$$
$$= A_{\bar{1}}(\lambda_1, \cdots, \lambda_n) h_n(\lambda_1, \lambda_2, \cdots, \lambda_n)_{\bar{1}, 2, \cdots, n, \bar{n}, \cdots, \bar{2}, 1},$$

$$(2.31) \, \mathcal{P}_{1,2}^- \cdot h_n(\lambda - 1, \lambda, \cdots, \lambda_n)_{1, \cdots, n, \bar{n}, \cdots \bar{1}} = -\frac{1}{2} s_{1,2}^{(1)} s_{\bar{1}, \bar{2}}^{(1)} h_{n-2}(\lambda_3, \cdots, \lambda_n)_{3, \cdots, n, \bar{n}, \cdots, \bar{3}},$$

$$(2.32) \, \mathcal{P}_{1, \bar{1}}^- \cdot h_n(\lambda_1, \cdots, \lambda_n)_{1, \cdots, n, \bar{n}, \cdots, \bar{1}} = (-1)^{n-1} \frac{1}{2} s_{1, \bar{1}}^{(1)} h_{n-1}(\lambda_2, \cdots, \lambda_n)_{2, \cdots, n, \bar{n}, \cdots, \bar{2}}.$$

In particular, (2.30) is a reduced form of the qKZ equation. Notice that, in contrast to the equation (2.24) for g_{2n}, the coefficients appearing in (2.29)and (2.30) are rational functions. This is a consequence of the properties $\rho(\lambda)\rho(-\lambda) = 1$, $\rho(\lambda - 1)\rho(\lambda) = -\lambda/(\lambda - 1)$.

An integral formula for the function h_n has been constructed in [12, 17]. Using that formula, we derive in Appendix B the following analytic properties of h_n.

Proposition 2.2. *The function $h_n(\lambda_1, \cdots, \lambda_n)$ satisfies the following:*

$$(2.33) \qquad\qquad h_n(\lambda_1, \cdots, \lambda_n) \text{ is meromorphic in } \lambda_1, \cdots, \lambda_n$$
$$\text{with at most simple poles at } \lambda_i - \lambda_j \in \mathbb{Z} \backslash \{0, \pm 1\},$$

$$(2.34) \qquad\qquad \text{for any } 0 < \delta < \pi \text{ we have}$$
$$\lim_{\substack{\lambda_1 \to \infty \\ \lambda_1 \in S_\delta}} h_n(\lambda_1, \cdots, \lambda_n) = (-1)^{n-1} \frac{1}{2} s_{1, \bar{1}}^{(1)} h_{n-1}(\lambda_2, \cdots, \lambda_n),$$
$$\text{where } S_\delta = \{\lambda \in \mathbb{C} \mid \delta < |\arg \lambda| < \pi - \delta\}.$$

As we shall show in the following sections, the properties (2.28)–(2.34) uniquely determine the functions h_n.

3. Recursion formula

In this section we state the main result of this paper: a recursion formula for the correlation functions. The main ingredient of the recursion is a transfer matrix with an auxiliary space of fractional dimension.

3.1. **Trace function.** We define "trace over a space of fractional dimension". By this we mean the unique $\mathbb{C}[x]$ linear map

$$\mathrm{Tr}_x \ : \ U(\mathfrak{sl}_2) \otimes \mathbb{C}[x] \longrightarrow \mathbb{C}[x]$$

such that for any non-negative integer k we have

$$(3.1) \qquad \mathrm{Tr}_{k+1}(A) = \mathrm{tr}_{V^{(k)}}\pi^{(k)}(A) \qquad (A \in U(\mathfrak{sl}_2)).$$

Here tr in the right hand side stands for the usual trace.

We list some properties of the trace function Tr_x.

$$(3.2) \qquad \mathrm{Tr}_x(AB) = \mathrm{Tr}_x(BA), \quad \mathrm{Tr}_x(1) = x,$$

$$(3.3) \qquad \mathrm{Tr}_x(A) = 0 \text{ if } A \text{ has non-zero weight},$$

$$(3.4) \qquad \mathrm{Tr}_x(e^{zH}) = \frac{\sinh(xz)}{\sinh z},$$

$$(3.5) \qquad \mathrm{Tr}_x\left(\left(\frac{H^2}{2} + H + 2FE\right)A\right) = \frac{x^2-1}{2}\mathrm{Tr}_x(A) \qquad (A \in U(\mathfrak{sl}_2) \otimes \mathbb{C}[x]).$$

By the generating series (3.4) the traces $\mathrm{Tr}_x(H^a)$ are known, and using (3.5) one can calculate $\mathrm{Tr}_x(H^a E^b F^c)$ inductively for all $a, b, c \geq 0$. We emphasize that $\mathrm{Tr}_x(A)$ is determined by the 'dimension' $\mathrm{Tr}_x(1) = x$ and the value of the Casimir operator; we have

$$(3.6) \qquad \mathrm{Tr}_x(A) = \mathrm{Tr}_x(A') \text{ if } \varpi_x(A) = \varpi_x(A'),$$

where ϖ_x is the projection

$$(3.7) \qquad \varpi_x : U(\mathfrak{sl}_2) \otimes \mathbb{C}[x] \to U(\mathfrak{sl}_2) \otimes \mathbb{C}[x]/I_x$$

and I_x signifies the two-sided ideal of $U(\mathfrak{sl}_2) \otimes \mathbb{C}[x]$ generated by $C - (x^2 - 1)/2$.

The following are simple consequences of these rules.

$$(3.8) \qquad \mathrm{Tr}_{-x}(A) = -\mathrm{Tr}_x(A),$$

$$(3.9) \qquad \mathrm{Tr}_x(A) - x\varepsilon(A) \in x(x^2 - 1)\mathbb{C}[x]$$
$$\text{where } \varepsilon : U(\mathfrak{sl}_2) \otimes \mathbb{C}[x] \to \mathbb{C}[x] \text{ stands for the counit},$$

$$(3.10) \qquad \text{The degree of } \mathrm{Tr}_x(H^a E^b F^c) \text{ is at most } m + 1 \ (m \text{ even})$$
$$\text{or } m \ (m \text{ odd}) \text{ where } m = a + b + c.$$

3.2. **Functions** $X^{[i,j]}$. We are now going to introduce our main object $X^{[i,j]}$.
For $1 \leq j \leq n$ (resp. $1 \leq i < j \leq n$), we set

$$(3.11) \qquad W^{[j]} = V_1 \otimes \cdots \overset{\overset{j}{\frown}}{\cdots} \otimes V_n \otimes V_{\bar{n}} \otimes \cdots \overset{\overset{\bar{j}}{\frown}}{\cdots} \otimes V_{\bar{1}} \,,$$

$$(3.12) \qquad W^{[i,j]} = V_1 \otimes \cdots \overset{\overset{i}{\frown}}{\cdots} \cdots \overset{\overset{j}{\frown}}{\cdots} \otimes V_n \otimes V_{\bar{n}} \otimes \cdots \overset{\overset{\bar{j}}{\frown}}{\cdots} \cdots \overset{\overset{\bar{i}}{\frown}}{\cdots} \otimes V_{\bar{1}} \,.$$

Define the monodromy matrices

$$(3.13) \qquad T^{[j]}(\lambda) = L_{\bar{1}}(\lambda - \lambda_1 - 1) \cdots \overset{\overset{\bar{j}}{\frown}}{\cdots} \cdots L_{\bar{n}}(\lambda - \lambda_n - 1)$$

$$\times \quad L_n(\lambda - \lambda_n) \cdots \overset{\overset{j}{\frown}}{\cdots} \cdots L_1(\lambda - \lambda_1)$$

$$(3.14) \qquad T^{[i,j]}(\lambda) = L_{\bar{1}}(\lambda - \lambda_1 - 1) \cdots \overset{\overset{\bar{i}}{\frown}}{\cdots} \cdots \overset{\overset{\bar{j}}{\frown}}{\cdots} \cdots L_{\bar{n}}(\lambda - \lambda_n - 1)$$

$$\times \quad L_n(\lambda - \lambda_n) \cdots \overset{\overset{j}{\frown}}{\cdots} \cdots \overset{\overset{i}{\frown}}{\cdots} \cdots L_1(\lambda - \lambda_1).$$

These are elements of $U(\mathfrak{sl}_2) \otimes \mathrm{End}(W^{[j]})$ and $U(\mathfrak{sl}_2) \otimes \mathrm{End}(W^{[i,j]})$, respectively.
Using the trace function Tr_x, we define the functions

$$X^{[i,j]}(\lambda_1, \cdots, \lambda_n) \in V_i \otimes V_{\bar{i}} \otimes V_j \otimes V_{\bar{j}} \otimes \mathrm{End}(W^{[i,j]}) \qquad (1 \leq i < j \leq n)$$

by the formula

(3.15)

$$X^{[i,j]}(\lambda_1, \cdots, \lambda_n) = \frac{1}{\lambda_{i,j}(\lambda_{i,j}^2 - 1) \prod_{p \neq i,j} \lambda_{i,p} \lambda_{j,p}}$$

$$\times R_{i,i-1}(\lambda_{i,i-1}) \cdots R_{i,1}(\lambda_{i,1}) R_{\overline{i-1},\bar{i}}(\lambda_{i-1,i}) \cdots R_{\bar{1}\bar{i}}(\lambda_{1,i}) \, \mathrm{Tr}_{\lambda_{i,j}} \left(T^{[i]} \left(\frac{\lambda_i + \lambda_j}{2} \right) \right)$$

$$\times R_{j,j-1}(\lambda_{j,j-1}) \cdots \widehat{R_{j,i}(\lambda_{j,i})} \cdots R_{j,1}(\lambda_{j,1}) R_{\overline{j-1},\bar{j}}(\lambda_{j-1,j}) \cdots \widehat{R_{\bar{i},\bar{j}}(\lambda_{i,j})} \cdots R_{\bar{1},\bar{j}}(\lambda_{1,j})$$

$$\times s^{(2)}_{(i,\bar{i}),(j,\bar{j})} \,.$$

One can think of $\mathrm{Tr}_{\lambda_{i,j}}(T^{[i]}(\frac{\lambda_i + \lambda_j}{2}))$ as a transfer matrix with "$\lambda_{i,j}$-dimensional auxiliary space".

The functions $X^{[i,j]}$ are not independent. From (3.15) and (3.18), $X^{[i,j]}$ with general i, j can be expressed in terms of one of them, e.g.,

$$(3.16) \qquad X^{[1,2]}(\lambda_1, \cdots, \lambda_n) = \frac{1}{\lambda_{1,2}(\lambda_{1,2}^2 - 1) \prod_{p=3}^{n} \lambda_{1,p} \lambda_{2,p}}$$

$$\times \mathrm{Tr}_{\lambda_{1,2}} \left(T^{[1]} \left(\frac{\lambda_1 + \lambda_2}{2} \right) \right) s^{(2)}_{(1,\bar{1}),(2,\bar{2})} \,,$$

as follows.

$$(3.17) \quad X^{[i,j]}(\lambda_1, \cdots, \lambda_n)$$

$$= R_{i,i-1}(\lambda_{i,i-1}) \cdots R_{i,1}(\lambda_{i,1}) R_{j,j-1}(\lambda_{j,j-1}) \cdots \widehat{R_{j,i}(\lambda_{j,i})} \cdots R_{j,1}(\lambda_{j,1})$$

$$\times \quad R_{\overline{i-1},\bar{i}}(\lambda_{i-1,i}) \cdots R_{\bar{1}\bar{i}}(\lambda_{1,i}) R_{\overline{j-1},\bar{j}}(\lambda_{j-1,j}) \cdots \widehat{R_{\bar{i},\bar{j}}(\lambda_{i,j})} \cdots R_{\bar{1},\bar{j}}(\lambda_{1,j})$$

$$\times \quad X^{[1,2]}(\lambda_i, \lambda_j, \lambda_1, \cdots, \widehat{\lambda_i}, \cdots, \widehat{\lambda_j}, \cdots, \lambda_n)_{i,j,1,\cdots,\hat{i},\cdots,\hat{j},\cdots,n,\bar{n},\cdots,\hat{\bar{j}},\cdots,\hat{\bar{i}},\cdots,\bar{1},\bar{j},\bar{i}}.$$

Nonetheless, for the description of the results, it is convenient to use all of $X^{[i,j]}$.

Let us list the main properties of $X^{[i,j]}$.

Transformation law: For an element σ of the symmetric group $\mathfrak{S}_n$, let us use the abbreviation

$$(X^{[i,j]})^\sigma = X^{[i,j]}(\lambda_{\sigma(1)}, \cdots, \lambda_{\sigma(n)})_{\sigma(1),\cdots,\sigma(n),\overline{\sigma(n)},\cdots,\overline{\sigma(1)}}.$$

Then the functions $X^{[i,j]} = (X^{[i,j]})^{\mathrm{id}}$ obey the transformation law

$$(3.18) \quad R_{k,k+1}(\lambda_{k,k+1}) R_{\overline{k+1},\bar{k}}(\lambda_{k+1,k}) X^{[i,j]}$$

$$= \begin{cases} (X^{[i,j]})^{(k,k+1)} R_{k,k+1}(\lambda_{k,k+1}) R_{\overline{k+1},\bar{k}}(\lambda_{k+1,k}) & (k \neq i, i-1, j, j-1), \\ (X^{[i,j+1]})^{(j,j+1)} & (k = j), \\ (X^{[i,j-1]})^{(j-1,j)} & (i < k = j-1), \\ (X^{[i-1,j]})^{(i-1,i)} & (k = i-1), \\ (X^{[i+1,j]})^{(i,i+1)} & (k = i < j-1), \\ (X^{[i,i+1]})^{(i,i+1)} & (i = k, j = i+1). \end{cases}$$

Pole structure: $\prod_{p(\neq i,j)} \lambda_{i,p} \lambda_{j,p} \cdot X^{[i,j]}(\lambda_1, \cdots, \lambda_n)$ is a polynomial.

Regularity at ∞: For each $1 \leq k \leq n$ and an $\mathfrak{sl}_2$-invariant vector $v \in (W^{[i,j]})^{\mathfrak{sl}_2}$, we have

$$(3.19) \quad X^{[i,j]}(\lambda_1, \cdots, \lambda_n)v = O(1) \qquad (\lambda_k \to \infty).$$

Difference equation: $X^{[1,2]}(\lambda_1, \cdots, \lambda_n)$ satisfies the difference equation

$$(3.20) \quad X^{[1,2]}(\lambda_1 - 1, \lambda_2, \cdots, \lambda_n)$$

$$= -\frac{(\lambda_{12} - 1)(\lambda_{12} + 1)}{\lambda_{12}(\lambda_{12} - 2)} A_{\bar{1}}(\lambda_1, \cdots, \lambda_n) P_{1,\bar{1}} X^{[1,2]}(\lambda_1, \lambda_2, \cdots, \lambda_n),$$

where $A_{\bar{1}}$ is defined in (2.27).

These properties will be proved in Section 4 (see Lemma 4.2–4.5 and Lemma 4.9).

3.3. **Main result.** We are now in a position to state the recursion formula which determines the functions $h_n(\lambda_1, \cdots, \lambda_n)$ starting from the initial condition

$$(3.21) \qquad\qquad h_0 = 1, \quad h_1(\lambda_1) = \frac{1}{2} s_{1,\bar{1}}^{(1)}.$$

The proofs for the statements in this subsection are given in Section 4.

In this subsection we write $X_n^{[i,j]}$ for $X^{[i,j]}$ to indicate the relevant number of sites n.

Theorem 3.1. *We have the following recursion formula.*

(3.22)

$$h_n(\lambda_1, \cdots, \lambda_n)_{1,\cdots,n,\bar{n},\cdots,\bar{1}}$$

$$= \sum_{j=2}^{n} Z_n^{[1,j]}(\lambda_1, \cdots, \lambda_n) \times h_{n-2}(\lambda_2, \cdots, \widehat{\lambda_j}, \cdots, \lambda_n)_{2,\cdots,\hat{j},\cdots,n,\bar{n},\cdots,\hat{\bar{j}},\cdots,\bar{2}}$$

$$+ \ (-1)^{n-1} \frac{1}{2} s_{1,\bar{1}}^{(1)} \cdot h_{n-1}(\lambda_2, \cdots, \lambda_n)_{2,\cdots,n,\bar{n},\cdots,\bar{2}}.$$

Here

(3.23)

$$\begin{aligned}
Z_n^{[1,j]}(\lambda_1, \cdots, \lambda_n) &= \oint_{\mathcal{C}} \frac{d\sigma}{2\pi i} \frac{\omega(\sigma - \lambda_j)}{\sigma - \lambda_1} X_n^{[1,j]}(\sigma, \lambda_2, \cdots, \lambda_n) \\
&= \omega(\lambda_{1,j}) X_n^{[1,j]}(\lambda_1, \lambda_2, \cdots, \lambda_n) \\
&\quad + \sum_{p(\neq 1,j)} \frac{\omega(\lambda_{p,j})}{\lambda_{p,1}} \mathrm{res}_{\sigma=\lambda_p} X_n^{[1,j]}(\sigma, \lambda_2, \cdots, \lambda_n),
\end{aligned}$$

where $\mathcal{C}$ is a simple closed curve encircling $\lambda_1, \cdots, \lambda_n$ anticlockwise, $\omega(\lambda)$ is given by (2.5), and $X_n^{[1,j]}(\lambda_1, \cdots, \lambda_n)$ is defined in (3.15).

In the last line of (3.23), we have used the fact that $X_n^{[1,j]}(\sigma, \lambda_2, \cdots, \lambda_n)$ has no poles at $\sigma = \lambda_j$.

Theorem 3.2. *The function $h_n(\lambda_1, \cdots, \lambda_n)$ has the structure*

(3.24)

$$h_n(\lambda_1, \cdots, \lambda_n) = \sum_{m=0}^{[n/2]} \sum_{I,J} \prod_{p=1}^{m} \omega(\lambda_{i_p} - \lambda_{j_p}) \, f_{n,I,J}(\lambda_1, \cdots, \lambda_n),$$

where $f_{n,I,J}(\lambda_1, \cdots, \lambda_n) \in V^{\otimes 2n}$ are rational functions, and $I = (i_1, \ldots, i_m)$, $J = (j_1, \ldots, j_m)$ run over sequences satisfying $I \cap J = \emptyset$, $i_1 < \cdots < i_m$, $1 \le i_p < j_p \le n$ $(1 \le p \le m)$. The representation of h_n in the above form is unique.

Theorem 3.3. *In the notation of Theorem 3.2, the rational functions $f_{n,I,J}(\lambda_1, \cdots, \lambda_n)$ are uniquely determined by the recursion relation*

(3.25)

$$\begin{aligned}
f_{n,iI',jJ'}(\lambda_1, \cdots, \lambda_n) &= X_n^{[i,j]}(\lambda_1, \cdots, \lambda_n) \\
&\quad \times f_{n-2,I',J'}(\lambda_1, \cdots, \widehat{\lambda_i}, \cdots, \widehat{\lambda_j}, \cdots, \lambda_n),
\end{aligned}$$

(3.26)

$$f_{n,\emptyset,\emptyset}(\lambda_1, \cdots, \lambda_n) = (-1)^{n(n-1)/2} \frac{1}{2^n} s_{1,\bar{1}}^{(1)} \cdots s_{n,\bar{n}}^{(1)}.$$

In addition, they enjoy the following properties.

(3.27) $f_{n,I,J}(\lambda_1, \cdots, \lambda_n)$ *is invariant under the action of* $\mathfrak{sl}_2$,

(3.28) $f_{n,I,J}(\lambda_1, \cdots, \lambda_{j+1}, \lambda_j, \cdots, \lambda_n)_{\cdots, j+1, j, \cdots, \cdots, \overline{j}, \overline{j+1}, \cdots}$

$$= R_{j,j+1}(\lambda_{j,j+1})R_{\overline{j+1},\overline{j}}(\lambda_{j+1,j})f_{n,\widetilde{I},\widetilde{J}}(\lambda_1, \cdots, \lambda_j, \lambda_{j+1}, \cdots, \lambda_n)_{\cdots, j, j+1, \cdots, \overline{j+1}, \overline{j}, \cdots},$$

where $\widetilde{I} = \sigma(I)$ *and* $\widetilde{J} = \sigma(J)$ ($\sigma = (j, j+1)$) *except for the following cases:*

if $i_m = j \in I$, $j_m = j + 1 \in J$ *for some* m, *we have* $\widetilde{I} = I, \widetilde{J} = J$;

if $i_m = j$, $i_{m+1} = j + 1$ *for some* m, *we have* $\widetilde{I} = I$ *and* $\widetilde{J}$ *is given by*

$$\widetilde{j}_l = \begin{cases} j_l & \text{if } l \neq m, m+1; \\ j_{m+1} & \text{if } l = m; \\ j_m & \text{if } l = m+1, \end{cases}$$

(3.29) $f_{n,I,J}(\lambda_1, \cdots, \lambda_n)$ *is regular at* ∞ *in each* λ_j,

(3.30) $f_{n,I,J}(\lambda_1, \cdots, \lambda_n)$ *has at most simple poles at* $\lambda_i - \lambda_j = 0$, *where*

$1 \leq i < j \leq n$, $i \in I$ *or* $j \in J$, $(i,j) \neq (i_p, j_p)(1 \leq p \leq m)$.

In general, $f_{n,I,J}$ is expressed as follows. Suppose $I = (i_1, \cdots, i_m)$, $J = (j_1, \cdots, j_m)$ and $\{1, \cdots, n\} \backslash I \cup J = \{k_1, \cdots, k_l\}$, where $k_1 < \cdots < k_l$, $n = 2m + l$. Denote the corresponding permutation by

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & \cdots & \cdots & \cdots & \cdots & n \\ i_1 & j_1 & \cdots & i_m & j_m & k_1 & \cdots & k_l \end{pmatrix} \quad \in \mathfrak{S}_n,$$

and let $\sigma = \sigma_{a_1} \circ \cdots \circ \sigma_{a_N}$ be a reduced decomposition into transpositions $\sigma_a = (a, a+1)$. We associate with σ the R matrices

$$R^\sigma = R_{b'_1, b_1}(\lambda_{b'_1, b_1}) \cdots R_{b'_N, b_N}(\lambda_{b'_N, b_N}),$$

$$\overline{R}^\sigma = R_{\overline{b_1}, \overline{b'_1}}(\lambda_{b_1, b'_1}) \cdots R_{\overline{b_N}, \overline{b'_N}}(\lambda_{b_N, b'_N}),$$

where $b_i = \sigma_{a_1} \cdots \sigma_{a_{i-1}}(a_i)$, $b'_i = \sigma_{a_1} \cdots \sigma_{a_{i-1}}(a_i + 1)$. Then

$$f_{n,I,J}(\lambda_1, \cdots, \lambda_n) = \frac{1}{D} R^\sigma \overline{R}^\sigma \tau_1 \tau_2 \cdots \tau_m \mathbf{s},$$
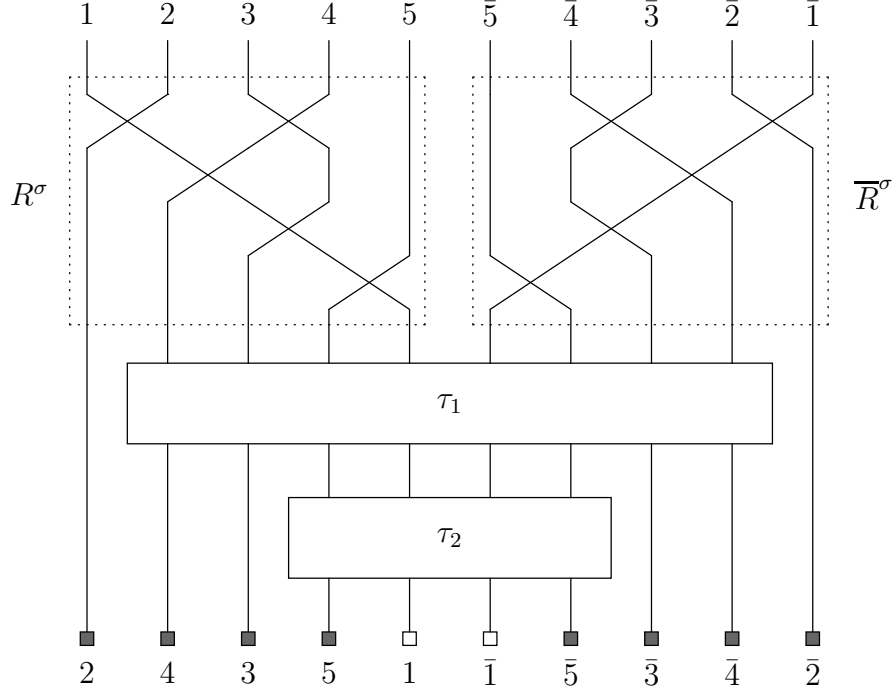
where

$$D = (-1)^{l(l-1)/2} 2^l \prod_{a<b} \lambda_{i_a, j_b} \lambda_{j_a, i_b} \lambda_{i_a, i_b} \lambda_{j_a, j_b} \prod_{a,c} \lambda_{i_a, k_c} \lambda_{j_a, k_c},$$

$\mathbf{s} = s^{(2)}_{(i_1, \overline{i_1}), (j_1, \overline{j_1})} \cdots s^{(2)}_{(i_m, \overline{i_m}), (j_m, \overline{j_m})} s^{(1)}_{k_1, \overline{k_1}} \cdots s^{(1)}_{k_l, \overline{k_l}}$ is the product of singlet vectors, and τ_a's are transfer matrices

$$\tau_a = \frac{1}{\lambda_{i_a, j_a}(\lambda^2_{i_a, j_a} - 1)} \mathrm{Tr}_{\lambda_{i_a, j_a}} \left(T_{n-2a+2}\left(\frac{\lambda_{i_a} + \lambda_{j_a}}{2}\right) \right),$$

$$T_{n-2a+2}(\lambda) = L_{\overline{\sigma(2a)}}(\lambda - \lambda_{\sigma(2a)} - 1) \cdots L_{\overline{\sigma(n)}}(\lambda - \lambda_{\sigma(n)} - 1)$$
$$\times L_{\sigma(n)}(\lambda - \lambda_{\sigma(n)}) \cdots L_{\sigma(2a)}(\lambda - \lambda_{\sigma(2a)}).$$

FIGURE 1. The case $n = 5$, $I = (23)$, $J = (45)$.

3.4. **Examples.** Let us write down the recursion relation in simple cases.
First let us take $n = 2$. Noting that

$$\mathrm{Tr}_x(AB) = \frac{1}{6}x(x^2 - 1)(A|B) \qquad (A, B \in \mathfrak{sl}_2)$$

and using the initial condition (3.21), we find

$$X^{[1,2]}(\lambda_1, \lambda_2) = \frac{1}{3}s^{(2)}_{(1,\bar{1}),(2,\bar{2})}.$$

Along with the initial condition (3.21), the recursion formula gives

$$h_2(\lambda_1, \lambda_2) = \frac{1}{3}\omega(\lambda_{1,2})s^{(2)}_{(1,\bar{1}),(2,\bar{2})} - \frac{1}{4}s^{(1)}_{1,\bar{1}}s^{(1)}_{2,\bar{2}}.$$

Next consider the case $n = 3$. We have

$$\mathrm{Tr}_x(ABC) = \frac{1}{12}x(x^2 - 1)([A, B]|C) \qquad (A, B, C \in \mathfrak{sl}_2).$$

Take a basis of the $\mathfrak{sl}_2$-invariants $(V^{\otimes 6})^{\mathfrak{sl}_2}$ as follows:

$$u_0 = s^{(1)}_{1,\bar{1}}s^{(1)}_{2,\bar{2}}s^{(1)}_{3,\bar{3}},$$
$$u_i = s^{(1)}_{i,\bar{i}}s^{(2)}_{(j,\bar{j}),(k,\bar{k})}, \quad (i, j, k = 1, 2, 3 \text{ are distinct}),$$
$$u_4 = u,$$

where u is the unique $\mathfrak{sl}_2$-invariant vector in $V^{(2)}_{(1,\bar{1})} \otimes V^{(2)}_{(2,\bar{2})} \otimes V^{(2)}_{(3,\bar{3})}$ with coefficient 1 in the component $v^{(2)}_0 \otimes v^{(2)}_1 \otimes v^{(2)}_2$.

After some calculation we find

$$X^{[1,2]}(\lambda_1, \lambda_2, \lambda_3)s_{3,\bar{3}}^{(1)} = \frac{1}{3}u_3 + \frac{1}{3\lambda_{1,3}\lambda_{2,3}}u_2 - \frac{\lambda_{1,2}}{6\lambda_{1,3}\lambda_{2,3}}u_4,$$

which gives

$$
\begin{aligned}
h_3(\lambda_1, \lambda_2, \lambda_3) &= \omega(\lambda_{1,2})\left(\frac{1}{6}u_3 + \frac{1}{6\lambda_{1,3}\lambda_{2,3}}u_2 - \frac{\lambda_{1,2}}{12\lambda_{1,3}\lambda_{2,3}}u_4\right) \\
&+ \omega(\lambda_{1,3})\left(\frac{1}{6}\left(1 - \frac{1}{\lambda_{1,2}\lambda_{2,3}}\right)u_2 + \frac{\lambda_{1,2} - \lambda_{2,3}}{12\lambda_{1,2}\lambda_{2,3}}u_4\right) \\
&+ \omega(\lambda_{2,3})\left(\frac{1}{6}u_1 + \frac{1}{6\lambda_{1,2}\lambda_{1,3}}u_2 + \frac{\lambda_{2,3}}{12\lambda_{1,2}\lambda_{1,3}}u_4\right) \\
&- \frac{1}{8}u_0.
\end{aligned}
$$

4. DERIVATION FROM THE qKZ EQUATION

The purpose of this section is to give a derivation of Theorem 3.1–3.3.

4.1. **Properties of** $X^{[i,j]}(\lambda_1, \cdots, \lambda_n)$. In this subsection, we study the properties of the rational function $X^{[i,j]}(\lambda_1, \cdots, \lambda_n)$.

We begin by showing an identity of L operators.

Lemma 4.1. *The following equality holds.*
(4.1)
$$\varpi_\lambda\left(L_2(\frac{\lambda}{2})L_{\bar{2}}(\frac{\lambda}{2} - 1)\right)s_{(1\bar{1}),(2\bar{2})}^{(2)} = \frac{1}{\lambda + 2}r_{(1,\bar{1}),\bar{2}}^{(2,1)}(\lambda + \frac{1}{2})\varpi_\lambda\left(L_{(2,\bar{2})}^{(2)}\left(\frac{\lambda + 1}{2}\right)\right)s_{(1,\bar{1}),(2,\bar{2})}^{(2)}.$$

Proof. We show (4.1) by projecting it to the symmetric and skew-symmetric subspaces of $V_2 \otimes V_{\bar{2}}$, separately. To simplify the notation, in what follows we shall write $A \sim B$ to indicate the equality $\varpi_\lambda(A) = \varpi_\lambda(B)$. Note that

(4.2)
$$C \sim \frac{\lambda^2 - 1}{2}.$$

From (2.11) we have

$$\mathcal{P}_{2,\bar{2}}^+ L_2(\frac{\lambda}{2})L_{\bar{2}}(\frac{\lambda}{2} - 1)s_{(1\bar{1}),(2\bar{2})}^{(2)} = L_{(2\bar{2})}^{(2)}(\frac{\lambda - 1}{2})s_{(1\bar{1}),(2\bar{2})}^{(2)}.$$

Hence, on the symmetric subspace, (4.1) reduces to
(4.3)
$$L_{(2\bar{2})}^{(2)}\left(\frac{\lambda - 1}{2}\right)s_{(1\bar{1}),(2\bar{2})}^{(2)} \sim \frac{1}{\lambda + 2}\left(\lambda + 1 + \frac{1}{2}\sum_a (S_a)_{(1\bar{1})}(S^a)_{(2\bar{2})}\right)L_{(2\bar{2})}^{(2)}\left(\frac{\lambda + 1}{2}\right)s_{(1\bar{1}),(2\bar{2})}^{(2)}.$$

To simplify the right hand side, notice the relation

(4.4)
$$\frac{1}{2}\sum_a S_a \otimes S^a = P^{(2)} - 3K^{(2)},$$

where $P^{(2)} \in \mathrm{End}_{\mathfrak{sl}_2}(V^{(2)} \otimes V^{(2)})$ is the permutation operator and $K^{(2)} \in \mathrm{End}_{\mathfrak{sl}_2}(V^{(2)} \otimes V^{(2)})$ is the projector onto the singlet subspace. More explicitly, we have

$$K^{(2)} = \frac{1}{3} s^{(2)} \otimes s^{(2)} \in (V^{(2)} \otimes V^{(2)})^{\otimes 2} \simeq \mathrm{End}(V^{(2)} \otimes V^{(2)}),$$

where we made the identification through the invariant bilinear form $(\cdot, \cdot) : V^{(2)} \otimes V^{(2)} \to \mathbb{C}$ normalized as $(v_0^{(2)}, v_2^{(2)}) = 1$.

By using (4.4), the right hand side of (4.3) can be rewritten as

$$\frac{1}{\lambda + 2}\left((\lambda + 1) L_{(2\bar{2})}^{(2)} \left(\frac{\lambda + 1}{2} \right) + L_{(2\bar{2})}^{(2)} \left(-\frac{\lambda + 3}{2} \right) - \mathrm{tr}_{(2\bar{2})} \left(L_{(2\bar{2})}^{(2)} \left(\frac{\lambda + 1}{2} \right) \right) \right) s_{(1\bar{1}),(2\bar{2})}^{(2)}.$$

In the second term, we used the crossing symmetry (2.14). The trace is evaluated as

$$(4.5) \qquad \mathrm{tr}_{(2\bar{2})} \left(L_{(2\bar{2})}^{(2)} \left(\frac{\lambda + 1}{2} \right) \right) \sim (\lambda + 1)(\lambda + 2).$$

Now one can check (4.3) directly by using (2.12).

Next let us project to the singlet subspace. The right hand side becomes

$$(4.6) \qquad \mathcal{P}_{2,\bar{2}}^-(\mathrm{RHS}) = \frac{1}{2(\lambda + 2)} \sum_a (S_a)_{(1,\bar{1})} \left((S^a)_{\bar{2}} - (S^a)_2 \right) L_{(2,\bar{2})}^{(2)} \left(\frac{\lambda + 1}{2} \right) s_{(1\bar{1}),(2\bar{2})}^{(2)}.$$

Using the matrix representation

$$L_{(2\bar{2})}^{(2)} \left(\frac{\lambda + 1}{2} \right) \sim \begin{pmatrix} \frac{(H + \lambda + 1)(H + \lambda + 3)}{4} & (H + \lambda + 3)F & F^2 \\ \frac{E(H + \lambda + 3)}{2} & \frac{(\lambda + 1)^2 - H^2}{2} & -\frac{F(H - \lambda - 3)}{2} \\ E^2 & -(H - \lambda - 3)E & \frac{(H - \lambda - 1)(H - \lambda - 3)}{4} \end{pmatrix},$$

we obtain

$$\mathcal{P}_{2,\bar{2}}^-(\mathrm{RHS}) \sim \left(-F(v_0^{(2)})_{(1\bar{1})} + \frac{1}{2} H(v_1^{(2)})_{(1\bar{1})} + E(v_2^{(2)})_{(1\bar{1})} \right) s_{2,\bar{2}}^{(1)}.$$

The left hand side becomes

$$\mathcal{P}_{2,\bar{2}}^-(\mathrm{LHS})$$
$$= \left(\mathcal{P}_{2,\bar{2}}^- L_{\bar{2}} \left(\frac{\lambda - 2}{2} \right) L_2 \left(\frac{\lambda}{2} \right) + \mathcal{P}_{2\bar{2}}^- \left[L_2 \left(\frac{\lambda}{2} \right), L_{\bar{2}} \left(\frac{\lambda - 2}{2} \right) \right] \right) s_{(1\bar{1}),(2\bar{2})}^{(2)}.$$

The first term is zero because of (2.15). Therefore, we have

$$(4.7) \qquad \mathcal{P}_{2,\bar{2}}^-(\mathrm{LHS}) \sim \mathcal{P}_{2,\bar{2}}^- \sum_{a,b} [S_a, S_b](S^a)_2 (S^b)_{\bar{2}} s_{(1\bar{1}),(2\bar{2})}^{(2)}.$$

Now the equality $\mathcal{P}_{2,\bar{2}}^-(\mathrm{LHS}) \sim \mathcal{P}_{2,\bar{2}}^-(\mathrm{RHS})$ is easy to see. $\square$

We have the following symmetry of $X^{[1,2]}$.

Lemma 4.2. *The function $X^{[1,2]}(\lambda_1, \cdots, \lambda_n)$ possesses the symmetry property*

$$(4.8) \qquad R_{2,1}(\lambda_{2,1}) R_{\bar{1},\bar{2}}(\lambda_{1,2}) X^{[1,2]}(\lambda_2, \lambda_1, \cdots, \lambda_n)_{2,1,\cdots,n,\bar{n},\cdots,\bar{1},\bar{2}}$$
$$= X^{[1,2]}(\lambda_1, \lambda_2, \cdots, \lambda_n)_{1,2,\cdots n,\bar{n},\cdots,\bar{2},\bar{1}}.$$

Proof. Recall the definition

(4.9) $X^{[1,2]}(\lambda_1, \cdots, \lambda_n)$

$$= \frac{1}{\lambda_{1,2}(\lambda_{1,2}^2 - 1) \prod_{p \neq 1,2} \lambda_{1,p} \lambda_{2,p}} \mathrm{Tr}_{\lambda_{1,2}} \left(T^{[1]} \left(\frac{\lambda_1 + \lambda_2}{2} \right) \right) s^{(2)}_{(1,\bar{1}),(2,\bar{2})},$$

where

(4.10) $T^{[1]}(\lambda) = L_{\bar{2}}(\lambda - \lambda_2 - 1) \cdots L_{\bar{n}}(\lambda - \lambda_n - 1) L_n(\lambda - \lambda_n) \cdots L_2(\lambda - \lambda_2).$

From this formula, we see that (4.8) is proven once we verify the relation

(4.11) $\varpi_\lambda \left(R_{2,1}(-\lambda) R_{\bar{1},\bar{2}}(\lambda) L_1 \left(-\frac{\lambda}{2} \right) L_{\bar{1}} \left(-\frac{\lambda}{2} - 1 \right) s^{(2)}_{(1\bar{1}),(2\bar{2})} \right)$

$$= \varpi_\lambda \left(L_2 \left(\frac{\lambda}{2} \right) L_{\bar{2}} \left(\frac{\lambda}{2} - 1 \right) s^{(2)}_{(1\bar{1}),(2\bar{2})} \right),$$

where we denoted $\lambda_{1,2}$ by λ. As before, let us write $A \sim B$ to mean $\varpi_\lambda(A) = \varpi_\lambda(B)$. Using Lemma 4.1, (4.11) can be rewritten as

$$R_{2,1}(-\lambda) R_{\bar{1},\bar{2}}(\lambda) \frac{1}{\lambda - 2} r^{(2,1)}_{(2,\bar{2}),\bar{1}} \left(-\lambda + \frac{1}{2} \right) L^{(2)}_{(1,\bar{1})} \left(\frac{-\lambda + 1}{2} \right) s^{(2)}_{(1\bar{1}),(2\bar{2})}$$

$$\sim -\frac{1}{\lambda + 2} r^{(2,1)}_{(1,\bar{1}),\bar{2}} \left(\lambda + \frac{1}{2} \right) L^{(2)}_{(2,\bar{2})} \left(\frac{\lambda + 1}{2} \right) s^{(2)}_{(1\bar{1}),(2\bar{2})},$$

which is further equivalent to

(4.12) $r^{(2,2)}_{(1\bar{1}),(2\bar{2})}(\lambda) L^{(2)}_{(2,\bar{2})} \left(\frac{\lambda + 1}{2} \right) s^{(2)}_{(1\bar{1}),(2\bar{2})}$

$$\sim (\lambda + 2)(\lambda + 1) L^{(2)}_{(1,\bar{1})} \left(\frac{-\lambda + 1}{2} \right) s^{(2)}_{(1\bar{1}),(2\bar{2})}$$

by using (2.17) and (2.19). The proof of (4.12) is similar to that of (4.3). We use the spectral decomposition

(4.13) $r^{(2,2)}(\lambda) = \lambda(\lambda + 1) I + 2(\lambda + 1) P^{(2)} - 6\lambda K^{(2)}$

which follows from (4.4) and the equality $(1/2) \sum_{a,b} (S_a S_b)_{(1,\bar{1})} \otimes (S^a S^b)_{(2,\bar{2})} = 2 - 6K^{(2)}$. Here $P^{(2)}, K^{(2)}$ have the same meaning as in (4.4).

Substituting (4.13) into the left hand side of (4.12) we obtain

$$r^{(2,2)}_{(1\bar{1}),(2\bar{2})}(\lambda) L^{(2)}_{(2,\bar{2})} \left(\frac{\lambda + 1}{2} \right) s^{(2)}_{(1\bar{1}),(2\bar{2})} = \left(\lambda(\lambda + 1) L^{(2)}_{(1,\bar{1})} \left(-1 - \frac{\lambda + 1}{2} \right) + \right.$$

$$\left. + 2(\lambda + 1) L^{(2)}_{(1,\bar{1})} \left(\frac{\lambda + 1}{2} \right) - 2\lambda(\lambda + 1)(\lambda + 2) \right) s^{(2)}_{(1\bar{1}),(2\bar{2})}.$$

The equality (4.12) can now be verified by a straightforward calculation.   □

Lemma 4.3. $X^{[i,j]}$ *obey the transformation rules* (3.18).

Proof. Except for the case $k = i = j - 1$, these are immediate consequences of the definition and the Yang-Baxter relation. The non-trivial case $k = i = j - 1$ follows from Lemma 4.2 and the Yang-Baxter relation.   □

The next lemma concerns the pole structure of $X^{[i,j]}$.

Lemma 4.4. $\prod_{p(\neq i,j)} \lambda_{i,p} \lambda_{j,p} \cdot X^{[i,j]}(\lambda_1, \cdots, \lambda_n)$ *is a polynomial.*

Proof. First let us consider $X^{[1,2]}$ given by (4.9), (4.10). We have to prove that the poles at $\lambda_1 = \lambda_2$, $\lambda_1 = \lambda_2 \pm 1$ are spurious. Because of the property (3.9) of Tr_x, it is enough to show that $\varepsilon\left(T^{[1]}((\lambda_1 + \lambda_2)/2)\right)$ is divisible by $\lambda_{1,2}^2 - 1$. This is indeed the case, since $\varepsilon\left(L_{\bar{2}}(\lambda_{1,2}/2 - 1)\right) = (\lambda_{1,2} - 1)/2$ and $\varepsilon\left(L_2(\lambda_{1,2}/2)\right) = (\lambda_{1,2} + 1)/2$.

Next consider the case $i = 1$ with j general. We use the relation between $X^{[1,j]}$ and $X^{[1,2]}$,

(4.14)
$$
\begin{aligned}
X^{[1,j]}(\lambda_1, \cdots, \lambda_n) &= R_{j,j-1}(\lambda_{j,j-1}) \cdots R_{j,2}(\lambda_{j,2}) \\
&\times R_{\overline{j-1},\bar{j}}(\lambda_{j-1,j}) \cdots\cdots R_{\bar{2},\bar{j}}(\lambda_{2,j}) \\
&\times X^{[1,2]}(\lambda_1, \lambda_j, \lambda_2, \cdots, \widehat{\lambda_j}, \cdots, \lambda_n)_{1,j,2,\cdots,\hat{j},\cdots,n,\bar{n},\cdots,\hat{\bar{j}},\cdots,\bar{2},\bar{j},\bar{1}}.
\end{aligned}
$$

We are to show that the poles $\lambda_i = \lambda_j \pm 1$ ($2 \le i < j$) contained in the R matrices are in fact spurious. Taking the residue of $X^{[1,j]}$ at $\lambda_i = \lambda_j + 1$ ($j > i$), we find the following fragment

$$
\mathrm{Tr}_{\lambda_{1,j}}\left(T^{[1]}\left(\frac{\lambda_1 + \lambda_j}{2}\right)\right) R_{j,j-1}(\lambda_{j,j-1}) \cdots R_{j,i+1}(\lambda_{j,i+1}) \mathcal{P}_{i,j}^- .
$$

Move the R-matrices to the left by using the Yang-Baxter relation, which permutes the L-operators in $T^{[1]}$. This will bring the L_j next to L_i. Thus we find

$$
L_j\left(\tfrac{\lambda_{1,j}}{2}\right) L_i\left(\tfrac{\lambda_{1,j}}{2} - 1\right) \mathcal{P}_{i,j}^- = 0,
$$

where we used the equation for the quantum determinant (2.15) and the fact that the 'dimension' of the auxiliary space equals to $\lambda_{1,j}$. The pole at $\lambda_i = \lambda_j - 1$ ($j > i$) can be treated similarly.

For general $X^{[i,j]}$, we use the symmetry (4.8) to obtain the two representations

$$
\begin{aligned}
X^{[i,j]}(\lambda_1, \cdots, \lambda_n) &= R_{i,i-1} \cdots R_{i,1} R_{\overline{i-1},\bar{i}} \cdots R_{\bar{1},\bar{i}} \\
&\times X^{[1,j]}(\lambda_i, \lambda_1, \cdots, \widehat{\lambda_i}, \cdots, \lambda_n)_{i,1,\cdots,\hat{i},\cdots,n,\bar{n},\cdots,\hat{\bar{i}},\cdots,\bar{1},\bar{i}} \\
&= R_{j,j-1} \cdots R_{j,1} R_{i+1,i} \cdots R_{j-1,i} R_{\overline{j-1},\bar{j}} \cdots R_{\bar{1},\bar{j}} R_{\bar{i},\overline{i+1}} \cdots R_{\bar{i},\overline{j-1}} \\
&\times X^{[1,j]}(\lambda_j, \lambda_1, \cdots, \widehat{\lambda_i}, \cdots, \lambda_{j-1}, \lambda_i, \lambda_{j+1}, \cdots, \lambda_n)_{j,1,\cdots,\hat{i},\cdots,j-1,i,j+1,\cdots,n,\bar{n},\cdots,\overline{j+1},\bar{i},\overline{j-1},\cdots,\hat{\bar{i}},\cdots,\bar{1},\bar{j}}.
\end{aligned}
$$

where $R_{i,j} = R_{i,j}(\lambda_{i,j})$ and $R_{\bar{j},\bar{i}} = R_{\bar{j},\bar{i}}(\lambda_{j,i})$. From these expressions, it is clear that the poles arising from the R matrices are spurious. $\qquad\square$

The following lemma shows that the function $X^{[i,j]}(\lambda_1, \cdots, \lambda_n)$ is regular at $\lambda_k = \infty$ when it is applied to a singlet vector.

Lemma 4.5. *For any vector* $v \in \left(W^{[i,j]}\right)^{\mathfrak{sl}_2}$ *and* $k = 1, \cdots, n$, *we have*

$$
X^{[i,j]}(\lambda_1, \cdots, \lambda_n) \cdot v = O(1) \qquad (\lambda_k \to \infty).
$$

Proof. Since $R_{i,j}(\lambda_{i,j})R_{\bar{j},\bar{i}}(\lambda_{j,i})$ is holomorphic and invertible at $\lambda_{i,j} = \infty$, it is enough to consider the case $(i,j) = (1,2)$. For $k \geq 3$, the assertion is clear from the definition (3.15). We will prove the case $k = 1$. The case $k = 2$ reduces to this case by the symmetry (4.8).

As before, we write $A \sim B$ for $\varpi_x(A) = \varpi_x(B)$. We show the following statement: For any $\mu_1, \cdots, \mu_{2N} \in \mathbb{C}$ and $u \in (V^{\otimes 2N})^{\mathfrak{sl}_2}$, there exist $c_{pqrs} \in \mathbb{C}$ such that

$$(4.15) \qquad L_1(\mu_1 + x/2) \cdots L_{2N}(\mu_{2N} + x/2)u$$

$$\sim \left(\sum_{p+q+r+s \leq N} c_{pqrs} H^p E^q F^r x^s \right) u.$$

The assertion then follows from the property (3.10) of Tr_x by taking $N = n - 2$.

We may assume that the μ_j's are mutually distinct. It is sufficient to prove (4.15) for $u = u_\sigma = \sigma u_1$, where $u_1 = s_{1,2}^{(1)} s_{3,4}^{(1)} \cdots s_{2N-1,2N}^{(1)}$ and $\sigma \in \mathfrak{S}_{2N}$. We use the induction on the length $\ell(\sigma)$. First consider the case $\sigma = \mathrm{id}$. Set $\Omega = \sum_{a=1}^{3} S_a \otimes \pi^{(1)}(S^a)$. Since $\Omega_2 s_{1,2}^{(1)} = -\Omega_1 s_{1,2}^{(1)}$ and $\Omega^2 + \Omega = \frac{1}{2} C \otimes \mathrm{id} \sim (x^2 - 1)/4$, we have

$$L_1(\mu_1 + x/2)L_2(\mu_2 + x/2)s_{1,2}^{(1)}$$

$$\sim \left(\mu_1\mu_2 + \frac{x+1}{2}(\mu_1 + \mu_2) + (\mu_2 - \mu_1 + 1)\Omega_1 + \frac{(x+1)^2}{4} - \frac{x^2 - 1}{4} \right) s_{1,2}^{(1)}.$$

We see that the term x^2 cancels. Hence we have (4.15) for $u = u_1$.

Suppose (4.15) is true for $u = u_\sigma$, and consider $\tau = (i, i+1)\sigma$ with $\ell(\tau) = \ell(\sigma)+1$. Using the Yang-Baxter relation, we have

$$P_{i,i+1}r_{i,i+1}(\mu_i - \mu_{i+1})L_1(\mu_1 + \frac{x}{2}) \cdots L_{2N}(\mu_{2N} + \frac{x}{2}) \cdot u_\sigma$$

$$= L_1(\mu_1 + \frac{x}{2}) \cdots \cdots L_i(\mu_{i+1} + \frac{x}{2})L_{i+1}(\mu_i + \frac{x}{2})$$

$$\cdots L_{2N}(\mu_{2N} + \frac{x}{2}) \cdot \left((\mu_i - \mu_{i+1})u_\tau + u_\sigma\right).$$

By the induction hypothesis, the left hand side and the second term in the right hand side have degree at most N when they are projected by ϖ_x. Therefore the statement is true also for $u = u_\tau$. $\qquad\square$

Using the properties of $X^{[i,j]}$, let us check that the pole structure of the recursion (3.22) agrees with that of h_n stated in (2.33). Namely, we have

Proposition 4.6. *Assume that the h_{n-1} and h_{n-2} satisfy the analyticity property as stated in (2.33). Then the same is true for h_n given by (3.22).*

Proof. We have shown in Lemma 4.4 that in the right hand side of (3.22), the only possible poles other than $\lambda_{i,j} \in \mathbb{Z}\setminus\{0, \pm1\}$ are $\lambda_i = \lambda_j$. We rewrite the recursion as

follows:

$$h_n(\lambda_1, \cdots, \lambda_n)_{1,\cdots,n,\bar{n},\cdots,\bar{1}}$$

$$= \sum_{j=2}^{n} \frac{1}{\prod\limits_{p \neq 1,j} \lambda_{jp} \prod_{i=2}^{j-1}(1-\lambda_{ij}^2)} \cdot \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{d\sigma}{\prod_{p=1}^{n}(\sigma - \lambda_p)} \cdot \frac{\omega(\sigma - \lambda_j)}{(\sigma - \lambda_j)^2 - 1}$$

$$\times \quad \mathrm{Tr}_{\sigma - \lambda_j}\Big(T^{[1]}\big(\frac{\sigma + \lambda_j}{2}\big)\Big) \cdot r_{j,j-1}(\lambda_{j,j-1}) \cdots r_{j,2}(\lambda_{j,2}) r_{\overline{j-1},\bar{j}}(\lambda_{j-1,j}) \cdots r_{\bar{2}\bar{j}}(\lambda_{2j})$$

$$\times \quad s^{(2)}_{(1\bar{1}),(j\bar{j})} h_{n-2}(\lambda_2, \cdots, \hat{\lambda}_j, \cdots, \lambda_n)_{2,\cdots,\hat{j},\cdots,n,\bar{n},\cdots,\hat{\bar{j}},\cdots,\bar{2}}$$

$$+ \quad (-1)^{n-1}\frac{1}{2} s^{(1)}_{(1\bar{1})} \cdot h_{n-1}(\lambda_2, \cdots, \lambda_n)_{2,\cdots,n,\bar{n},\cdots,\bar{2}}.$$

Here the contour $\mathcal{C}$ goes around $\lambda_1, \cdots, \lambda_n$. From this expression, we see that the poles at $\lambda_{1p} = 0$ are spurious.

Consider the pole at $\lambda_k = \lambda_j$ with $k > j > 1$. We have

$$\mathrm{res}_{\lambda_k = \lambda_j} h_n(\lambda_1, \cdots, \lambda_n)$$

$$= \frac{1}{\prod\limits_{p \neq 1,j,k} \lambda_{jp}} \frac{1}{2\pi i} \oint \frac{d\sigma}{\prod_{p=1}^{n}(\sigma - \lambda_p)} \cdot \frac{\omega(\sigma - \lambda_j)}{(\sigma - \lambda_j)^2 - 1} \mathrm{Tr}_{\sigma - \lambda_j}\Big(T^{[1]}\big(\frac{\sigma + \lambda_j}{2}\big)\Big)$$

$$\{-R_{j,j-1}(\lambda_{j,j-1}) \cdots R_{j,2}(\lambda_{j,2}) R_{\overline{j-1},\bar{j}}(\lambda_{j-1,j}) \cdots R_{\bar{2}\bar{j}}(\lambda_{2j})$$

$$\times s^{(2)}_{(1\bar{1}),(j\bar{j})} h_{n-2}(\lambda_2, \cdots, \hat{\lambda}_j, \cdots, \lambda_n)_{2,\cdots,\hat{j},\cdots,n,\bar{n},\cdots,\hat{\bar{j}},\cdots,\bar{2}}\Big|_{\lambda_k = \lambda_j}$$

$$+R_{k,k-1}(\lambda_{j,k-1}) \cdots R_{k,j+1}(\lambda_{j,j+1}) R_{\overline{k-1},\bar{k}}(\lambda_{k-1,j}) \cdots R_{\overline{j+1},\bar{k}}(\lambda_{j+1,j}) P_{j,k} P_{\bar{j},\bar{k}}$$

$$\times R_{k,j-1}(\lambda_{j,j-1}) \cdots R_{k,2}(\lambda_{j,2}) R_{\overline{j-1},\bar{k}}(\lambda_{j-1,j}) \cdots R_{\bar{2}\bar{k}}(\lambda_{2j})$$

$$\times s^{(2)}_{(1\bar{1}),(k\bar{k})} h_{n-2}(\lambda_2, \cdots, \hat{\lambda}_k, \cdots, \lambda_n)_{2,\cdots,\hat{k},\cdots,n,\bar{n},\cdots,\hat{\bar{k}},\cdots,\bar{2}}\}$$

In the second term inside the braces, move $P_{j,k} P_{\bar{j},\bar{k}}$ to the right, then apply symmetry (2.29). After that the second term cancels the first one. $\qquad \square$

4.2. **Proof of the Ansatz.** In this subsection we prove the Ansatz (2.4).

Proof of Theorem 3.2. The assertion is clear for $n = 0, 1$. By induction, assume that it is true for $h_{n'}$ with $n' < n$. We use the recursion formula in the form

(4.16) $\quad h_n(\lambda_1, \cdots, \lambda_n)_{1,\cdots,n,\bar{n},\cdots,\bar{1}}$

$$= \sum_{j=2}^{n} \omega(\lambda_{1,j}) X^{[1,j]}(\lambda_1, \cdots, \lambda_n) h_{n-2}(\lambda_2, \cdots, \widehat{\lambda}_j, \cdots, \lambda_n)_{2,\cdots,\hat{j},\cdots,n,\bar{n},\cdots,\hat{\bar{j}},\cdots,\bar{2}}$$

$$+ \sum_{\substack{2 \leq j,k \leq n \\ j \neq k}} \frac{\omega(\lambda_{k,j})}{\lambda_{k,1}} \mathrm{res}_{\sigma = \lambda_k} X^{[1,j]}(\sigma, \lambda_2, \cdots, \lambda_n)$$

$$\times h_{n-2}(\lambda_2, \cdots, \widehat{\lambda}_j, \cdots, \lambda_n)_{2,\cdots,\hat{j},\cdots,n,\bar{n},\cdots,\hat{\bar{j}},\cdots,\bar{2}}$$

$$+(-1)^{n-1}\frac{1}{2} s^{(1)}_{1,\bar{1}} h_{n-1}(\lambda_2, \cdots, \lambda_n)_{2,\cdots,n,\bar{n},\cdots,\bar{2}}.$$

By the induction hypothesis, the right hand side becomes a linear combination of the form

$$\prod_{p=1}^{m} \omega(\lambda_{i_p} - \lambda_{j_p}) \, f_{n,I,J}(\lambda_1, \cdots, \lambda_n)$$

with some rational function $f_{n,I,J}(\lambda_1, \cdots, \lambda_n)$ and $1 \le i_1 \le \cdots \le i_m \le n$, $1 \le i_p < j_p \le n$. Since the product of the $\omega(\lambda)$'s are linearly independent over the field $\mathbb{C}(\lambda_1, \cdots, \lambda_n)$ of rational functions, this representation is unique. If there is a term for which $i_1, \cdots, i_m, j_1, \cdots, j_m$ are not distinct, then the symmetry relation (2.29) of h_n entails that there is also a term such that the index 1 appears more than once in $i_1, \cdots, i_m, j_1, \cdots, j_m$. However, terms with $i_1 = 1$ arise only from the first term of (4.16) and the corresponding indices are distinct. Therefore we have (3.24).   □

Proof of Theorem 3.3.

From the above proof, the recursion relations (3.25), (3.26) for $f_{n,I,J}$ are clear. Since the rational coefficients in (3.24) are unique, the $\mathfrak{sl}_2$-invariance (3.27) of $f_{n,I,J}$ follows from that of h_n. Similarly, the exchange symmetry (3.28) follows from the symmetry (2.29) of h_n by comparing the coefficients of the ω's. The regularity (3.29) at ∞ is a consequence of Lemma 4.5 and the $\mathfrak{sl}_2$-invariance (3.27). Finally the pole structure (3.30) follows from the recursion relations (3.25), (3.26) and Lemma 4.4.   □

4.3. **Calculation of the residues.** In this subsection, we compute the residues of $h_n(\lambda_1, \cdots, \lambda_n)$ with respect to λ_1. Our goal is Proposition 4.10 and 4.11. Since the calculation is long, we outline the proof before starting.

We calculate the residues of the meromorphic function $h_n(\lambda_1, \ldots, \lambda_n)$ at its poles. The poles are located at the points of the form $\lambda_{1,j} = k \in \mathbb{Z}\backslash\{0, \pm 1\}$. Then, we show that the right hand side of (3.22) has the same residues at these poles:

(4.17) $\mathrm{res}_{\lambda_{1j}=k} h_n(\lambda_1, \cdots, \lambda_n)_{1 \cdots n \bar{n} \cdots \bar{1}}$

$\quad = \mathrm{res}_{\lambda_{1j}=k}\{\omega(\lambda_{1,j}) X^{[1,j]}(\lambda_1, \cdots, \lambda_n)\} h_{n-2}(\lambda_2, \cdots, \widehat{\lambda_j}, \cdots, \lambda_n)_{2, \cdots, \hat{j}, \cdots, n, \bar{n}, \cdots, \hat{\bar{j}}, \cdots, \bar{2}}.$

A remarkable fact is that the residues at infinitely many points can be recursively described by using a single analytic function $\omega(\lambda_{1,j}) X^{[1,j]}(\lambda_1, \cdots, \lambda_n)$. Thus, we show that the difference between the left and right hand sides is an entire function of λ_1. Showing that it vanishes at $\lambda_1 \to \infty$, we obtain the equality.

The calculation of residues goes as follows. Suppose $j = 2$. Using (2.30) repeatedly, we can express $h_n(\lambda_1 - k - 1, \lambda_2, \ldots, \lambda_n)$ in terms of several A_1 and $A_{\bar{1}}$ with shifted arguments acting on $h_n(\lambda_1 - 1, \lambda_2, \ldots, \lambda_n)$. Using this expression, one can calculate the residue of $h_n(\lambda_1, \lambda_2, \ldots, \lambda_n)$ at the pole $\lambda_1 = \lambda_2 - k - 1$. If $k = 2$, for example, we have

$$\begin{aligned}
& \mathrm{res}_{\lambda_1 = \lambda_2 - 3} h_n(\lambda_1, \lambda_2, \ldots, \lambda_n) \\
= \ & \mathrm{res}_{\lambda_1 = \lambda_2} h_n(\lambda_1 - 3, \lambda_2, \ldots, \lambda_n) \\
= \ & \mathrm{res}_{\lambda_1 = \lambda_2}\{A_{\bar{1}}(\lambda_1 - 2, \ldots) A_1(\lambda_1 - 1, \ldots) h_n(\lambda_1 - 1, \ldots)\}
\end{aligned}$$

Because of (2.33), the pole at $\lambda_1 = \lambda_2$ in the last expression comes only from one of the R matrices in $A_1(\lambda_1 - 1, \ldots)$, i.e., $R_{1,2}(\lambda_{1,2} - 1)$. Its residue at $\lambda_{1,2} = 0$ is proportional to the projector $\mathcal{P}_{1,2}^-$, therefore one can use (2.31). In this way, for any k, we get a certain expression for $\mathrm{res}_{\lambda_1 = \lambda_2 - k - 1} h_n(\lambda_1, \ldots, \lambda_n)$; a bunch of R matrices acting on $s_{\gamma 2}^{(1)} s_{\overline{\gamma} \overline{2}}^{(1)} h_{n-2}(\lambda_3, \ldots, \lambda_n)$, where $\gamma = 1$ (k: even) or $\bar{1}$ (k: odd).

Next we rewrite the product of R matrices by using the Yang-Baxter relation. They act on the spaces V_α indexed by $\alpha = 1, \ldots, n, \bar{n}, \ldots, \bar{1}$. Three groups of indexes, $\{1, \bar{1}\}$, $\{2, \bar{2}\}$ and $\{3, \bar{3}, \ldots, n, \bar{n}\}$, play separate roles in the product. An R matrix $R_{i,j}$ acts on two components V_i and V_j. We call the first component V_i the auxiliary space and the second component V_j the quantum space. For any R matrix contained in A_α, the auxiliary space is indexed by 1 or $\bar{1}$, and the quantum space by the other two groups. The second and the third groups are distinct when the residue at $\lambda_{1,2} = 0$ is calculated and the vector $s_{\gamma 2}^{(1)} s_{\overline{\gamma} \overline{2}}^{(1)} h_{n-2}(\lambda_3, \ldots, \lambda_n)$ is created.

Our goal is to rewrite the product of R matrices by means of the transfer matrix $t^{(k)}(\lambda) = \mathrm{tr}_{V^{(k)}} \pi^{(k)} \left(T^{[1]}(\lambda) \right) \in \mathrm{End}(W^{[1]})$, where $T^{[1]}(\lambda)$ is given by (3.13), and $W^{[1]}$ by (3.11). Let us compare the product of transfer matrices $t^{(1)}(\lambda_1 - k + 1) \cdots t^{(1)}(\lambda_1)$ and the product $A_{\bar{1}}(\lambda_1 - k, \ldots) A_1(\lambda_1 - k + 1, \ldots) \cdots A_\gamma(\lambda_1 - 1, \ldots)$. In the former, the matrix product is taken only on the quantum spaces indexed by $2, \ldots, n, \bar{n} \ldots, \bar{2}$; in the latter not only on the quantum spaces but also the auxiliary spaces indexed by 1 and $\bar{1}$.

In a graphical representation of the product, we draw vertical lines for quantum spaces and horizontal lines for auxiliary spaces. In the former product we have k horizontal lines, and they form closed circles reflecting the trace; in the latter we have only two horizontal lines corresponding to 1 and $\bar{1}$. They form spirals. We can rewrite the latter as a trace by introducing k auxiliary spaces indexed by $\alpha_1, \ldots, \alpha_k$. We do the following procedure. We cut the horizontal lines in front of the vertical line corresponding to $\bar{3}$. We obtain k separate horizontal lines. We rename these lines by $\alpha_1, \ldots, \alpha_k$. Namely, we replace the spaces V_1 and $V_{\bar{1}}$ on these lines by $V_{\alpha_1}, \ldots, V_{\alpha_k}$. We recover the original product by taking the traces on these new auxiliary spaces.

After some manipulation using the Yang-Baxter relation, the whole expression comes in two parts: the fused monodromy operator

$$\pi_{(\alpha_1, \cdots \alpha_k)}^{(k)} \left(T^{[1,2]}(\lambda_2 - (k+1)/2) \right)$$

and the rest. The latter acts on the vector $s_{\gamma 2}^{(1)} s_{\overline{\gamma} \overline{2}}^{(1)}$. This action can be rewritten as

$$\pi_{(\alpha_1, \cdots \alpha_k)}^{(k)} \left(L_2(-\frac{k+1}{2}) L_{\bar{2}}(-\frac{k+3}{2}) \right) s_{(1\bar{1}),(2\bar{2})}^{(2)}.$$

Combining these two expressions inside the trace $\mathrm{tr}_{\alpha_1, \ldots, \alpha_k}$ we obtain (4.17) for $j = 2$ and $k \leq -3$. The calculation of the residues at the rest of the poles can be done by using symmetries.

Let us set

(4.18) $t_\alpha(\lambda_2) = r_{\alpha \bar{3}}(\lambda_{23} - 1) \cdots r_{\alpha \bar{n}}(\lambda_{2n} - 1) r_{\alpha n}(\lambda_{2n}) \cdots r_{\alpha 3}(\lambda_{23}),$

and recall that

$$T^{[1,2]}(\lambda) = L_{\bar{3}}(\lambda - \lambda_3 - 1) \cdots L_{\bar{n}}(\lambda - \lambda_n - 1) L_n(\lambda - \lambda_n) \cdots L_3(\lambda - \lambda_3),$$
$$T^{[1]}(\lambda) = L_{\bar{2}}(\lambda - \lambda_2 - 1) T^{[1,2]}(\lambda) L_2(\lambda - \lambda_2).$$

By using (2.17), we obtain

$$(4.19) \qquad t_{\alpha_k}(\lambda_2 - k) \cdots t_{\alpha_1}(\lambda_2 - 1) \mathcal{P}^+_{\alpha_1, \cdots, \alpha_k}$$

$$= \frac{\prod_{j=3}^n \prod_{l=1}^k \{(\lambda_{2j} - l)^2 - 1\}}{\prod_{j=3}^n \lambda_{2j}(\lambda_{2j} - k - 1)} \pi^{(k)}_{(\alpha_1, \cdots, \alpha_k)} \left(T^{[1,2]}(\lambda_2 - \frac{k+1}{2}) \right).$$

Here $\mathcal{P}^+_{\alpha_1, \cdots, \alpha_k}$ stands for the projector onto the completely symmetric tensors.

Now we start the calculation. We rewrite A_α given by (2.27):

$$(4.20) \qquad A_\alpha(\lambda_1, \lambda_2, \ldots, \lambda_n) = \frac{-1}{\prod_{j=2}^n(\lambda_{1j}^2 - 1)} r_{\alpha\bar{2}}(\lambda_{12} - 1) t_\alpha(\lambda_1) r_{\alpha 2}(\lambda_{12}),$$

In this form it is easy to see

$$(4.21) \qquad \mathrm{res}_{\lambda_{12}=-1} A_\alpha(\lambda_1, \ldots, \lambda_n) = \frac{-1}{\prod_{j=3}^n \lambda_{2j}(\lambda_{2j} - 2)} r_{\alpha\bar{2}}(-2) t_\alpha(\lambda_2 - 1) \mathcal{P}^-_{\alpha 2}.$$

Using (2.30), (4.21) and (2.31) we obtain

$$\mathrm{res}_{\lambda_{12}=-2} h_n(\lambda_1, \lambda_2, \ldots, \lambda_n)_{12 \cdots n\bar{n} \cdots \bar{2}\bar{1}}$$

$$= \frac{1}{2 \prod_{j=3}^n \lambda_{2,j}(\lambda_{2,j} - 2)} r_{\bar{1}\bar{2}}(-2) t_{\bar{1}}(\lambda_2 - 1) s^{(1)}_{\bar{1}\bar{2}} s^{(1)}_{12} h_{n-2}(\lambda_3, \ldots, \lambda_n)_{3 \cdots n\bar{n} \cdots \bar{3}}.$$

Now we consider the residue at $\lambda_{12} = -k - 1$ where $k \geq 2$. We set

$$(4.22) \qquad \gamma = \begin{cases} 1 & \text{if } k \text{ is even;} \\ \bar{1} & \text{if } k \text{ is odd.} \end{cases}$$

The only poles in (4.20) are $\lambda_{1j} = \pm 1$. By using (2.30) repeatedly we obtain

$$(4.23) \qquad \mathrm{res}_{\lambda_{12}=-k-1} h_n(\lambda_1, \lambda_2, \ldots, \lambda_n)_{12 \cdots n\bar{n} \cdots \bar{2}\bar{1}} = X_k h_{n-2}(\lambda_3, \ldots, \lambda_n)_{3 \cdots n\bar{n} \cdots \bar{3}},$$

for $k \geq 1$, where

$$(4.24) \ X_k = \frac{1}{2 \prod_{j=3}^n \lambda_{2,j}(\lambda_{2,j} - 2)} A_{\bar{1}}(\lambda_2 - k, \lambda_2, \ldots, \lambda_n) A_1(\lambda_2 - k + 1, \lambda_2, \ldots, \lambda_n)$$

$$\times \cdots \times A_{\bar{\gamma}}(\lambda_2 - 2, \lambda_2, \ldots, \lambda_n) r_{\gamma\bar{2}}(-2) t_\gamma(\lambda_2 - 1) s^{(1)}_{\gamma 2} s^{(1)}_{\bar{\gamma}\bar{2}}.$$

If $k \geq 2$, using (4.20) we can rewrite (4.24) as

$$(4.25) \quad X_k = \frac{(-1)^{k+1}}{(k-1)!(k+1)! \prod_{l=1}^k \prod_{j=3}^n((\lambda_{2,j} - l)^2 - 1)} r_{\bar{1}\bar{2}}(-k - 1) r_{1\bar{2}}(-k) \bullet Y_k$$

where

$$(4.26) \ Y_k = t_{\bar{1}}(\lambda_2 - k) r_{\bar{1}2}(-k) r_{\bar{1}\bar{2}}(-k + 1) t_1(\lambda_2 - k + 1) r_{12}(-k + 1) r_{1\bar{2}}(-k + 2)$$

$$\times \cdots \times t_\gamma(\lambda_2 - 3) r_{\gamma 2}(-3) r_{\gamma\bar{2}}(-2) t_{\bar{\gamma}}(\lambda_2 - 2) r_{\bar{\gamma}2}(-2) t_\gamma(\lambda_2 - 1) s^{(1)}_{\gamma 2} s^{(1)}_{\bar{\gamma}\bar{2}}.$$

We use the following simple identity, which follows from (2.15).

$$(4.27) \qquad\qquad r_{12}(-1)r_{13}(-2)r_{23}(-1) = 0.$$

Lemma 4.7. *We can insert $\mathcal{P}^+_{1\bar{1}}$ at the position $\bullet$ in (4.25).*

Proof. Since $\mathcal{P}^+_{1\bar{1}} + \mathcal{P}^-_{1\bar{1}} = \mathrm{id}_{1\bar{1}}$ and $-2\mathcal{P}^-_{1\bar{1}} = r_{1\bar{1}}(-1)$, it is enough to show that if we insert $r_{1\bar{1}}(-1)$ at the position $\bullet$ then (4.25) vanishes. Note that $s^{(1)} = -\frac{1}{2}r(-1)s^{(1)}$. If $\gamma = 1$, by using the Yang-Baxter relation, one can find $r_{1\bar{1}}(-1)r_{\bar{1}2}(-2)r_{12}(-1)$ therein; if $\gamma = \bar{1}$, $r_{1\bar{1}}(-1)r_{1\bar{2}}(-2)r_{\bar{1}2}(-1)$ instead. $\qquad\square$

Then, by using (2.19) we have

$$(4.28) \qquad X_k = \frac{(-1)^k k}{(k-1)!(k+1)! \prod_{l=1}^{k}\prod_{j=3}^{n}((\lambda_{2,j}-l)^2 - 1)}r^{(2,1)}_{(1\bar{1}),\bar{2}}(-k - \tfrac{1}{2})Y_k.$$

Now we use the identity

$$(4.29) \qquad\qquad t_1(\lambda - j) = \mathrm{tr}_{V_\alpha}\left(t_\alpha(\lambda - j)P_{\alpha,1}\right).$$

We can rewrite (4.26) as

$$(4.30) \qquad\begin{aligned} Y_k = \mathrm{tr}_{\alpha_1\cdots\alpha_k}\Big(& t_{\alpha_k}(\lambda_2 - k)\cdots t_{\alpha_1}(\lambda_2 - 1)\bullet \\ & \times r_{\alpha_k 2}(-k)\cdots r_{\alpha_3 2}(-3)r_{\alpha_k\bar{2}}(-k+1)\cdots r_{\alpha_3\bar{2}}(-2) \\ & \times r_{\alpha_2 2}(-2)P_{\alpha_k\bar{1}}P_{\alpha_{k-1}1}\cdots P_{\alpha_2\bar{\gamma}}P_{\alpha_1\gamma}\bullet\Big)s^{(1)}_{\gamma 2}s^{(1)}_{\bar{\gamma}\bar{2}}, \end{aligned}$$

where $\mathrm{tr}_{\alpha_1,\cdots,\alpha_k}$ stands for the trace over $V_{\alpha_1}\otimes\cdots\otimes V_{\alpha_k}$.

Lemma 4.8. *We can insert $\mathcal{P}^+_{\alpha_k\cdots\alpha_1}$ at the position $\bullet$ in (4.30) for both places.*

Proof. We prove the assertion for the first $\bullet$. Then, the assertion for the second $\bullet$ follows from the cyclicity of the trace.

Define an element of $\mathrm{End}(W)$,

$$(4.31) \qquad \tilde{Y}_k = \mathrm{tr}_{\alpha_k,\ldots,\alpha_1}\left((t_k)_{\alpha_k,\ldots,\alpha_1}r_{\alpha_k,\ldots,\alpha_1;2,\bar{2}}P_{\alpha_k\bar{1}}P_{\alpha_{k-1}1}\cdots P_{\alpha_2\bar{\gamma}}P_{\alpha_1\gamma}\right)$$

where

$$(4.32) \qquad (t_k)_{\alpha_k,\ldots,\alpha_1} = t_{\alpha_k}(\lambda_2 - k)\cdots t_{\alpha_1}(\lambda_2 - 1),$$

$$(4.33) \qquad r_{\alpha_k,\ldots,\alpha_1;2,\bar{2}} = r_{\alpha_k 2}(-k)\cdots r_{\alpha_1 2}(-1)r_{\alpha_k\bar{2}}(-k+1)\cdots r_{\alpha_2\bar{2}}(-1).$$

Let t_k^{sym} be the symmetrization of t_k,

$$(4.34) \quad t_k^{\mathrm{sym}}(v_{\tau_k}\otimes\cdots\otimes v_{\tau_1}) = \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k} t(\lambda_2 - k)^{\tau'_k}_{\tau_{\sigma(k)}}\cdots t(\lambda_2 - 1)^{\tau'_1}_{\tau_{\sigma(1)}}v_{\tau'_k}\otimes\cdots\otimes v_{\tau'_1}.$$

By the definition the matrix element $(t_k^{\mathrm{sym}})^{\tau'_k\cdots\tau'_1}_{\tau_k\cdots\tau_1}$ depends on $\tau_1,\ldots,\tau_k$ only by $\tau_k + \cdots + \tau_1 = i$. We write it $(t_k^{\mathrm{sym}})^{\tau'_k\cdots\tau'_1}_i$. We have the following symmetry of t_k^{sym}.

$$(4.35) \qquad\qquad (t_k^{\mathrm{sym}})^{\tau'_k\cdots\tau'_1}_i = (t_k^{\mathrm{sym}})^{\tau'_{\sigma(k)}\cdots\tau'_{\sigma(1)}}_i.$$

Let $\tilde{Y}'_k$ denote the matrix $\tilde{Y}_k$ in which t_k is replaced with t_k^{sym}. We are to show that $\tilde{Y}_k = \tilde{Y}'_k$. We use the following equivalent definition of $\tilde{Y}_k$ (and a similar one for $\tilde{Y}'_k$).

$$(4.36) \qquad \tilde{Y}_k(v_{\varepsilon_1} \otimes v_{\varepsilon_{\bar{1}}}) \;=\; \sum_{\varepsilon'_1, \varepsilon'_{\bar{1}}} \sum_{\substack{\tau_1, \ldots, \tau_k \\ \tau'_1, \ldots, \tau'_{k-2}}} ((t_k)_{\alpha_k, \ldots, \alpha_1})^{\varepsilon'_{\bar{1}} \varepsilon'_1 \tau'_{k-2} \cdots \tau'_1}_{\tau_k \tau_{k-1} \tau_{k-2} \cdots \tau_1}$$
$$\times\; \left(r_{\alpha_k, \ldots, \alpha_1; 2, \bar{2}} \right)^{\tau_k \tau_{k-1} \tau_{k-2} \cdots \tau_1}_{\tau'_{k-2} \cdots \tau'_1 \varepsilon_{\bar{\gamma}} \varepsilon_{\gamma}} (v_{\varepsilon'_1} \otimes v_{\varepsilon'_{\bar{1}}}).$$

We will show the following equality for each i.

$$(4.37) \qquad \sum_{\substack{\tau_1, \ldots, \tau_k \\ \tau_1 + \cdots + \tau_k = i \\ \tau'_1, \ldots, \tau'_{k-2}}} ((t_k)_{\alpha_k, \ldots, \alpha_1})^{\varepsilon'_{\bar{1}} \varepsilon'_1 \tau'_{k-2} \cdots \tau'_1}_{\tau_k \tau_{k-1} \tau_{k-2} \cdots \tau_1} \left(r_{\alpha_k, \ldots, \alpha_1; 2, \bar{2}} \right)^{\tau_k \tau_{k-1} \tau_{k-2} \cdots \tau_1}_{\tau'_{k-2} \cdots \tau'_1 \varepsilon_{\bar{\gamma}} \varepsilon_{\gamma}}$$
$$=\; \sum_{\substack{\tau_1, \ldots, \tau_k \\ \tau_1 + \cdots + \tau_k = i}} \left\{ \sum_{\tau'_1, \ldots, \tau'_{k-2}} (t_k^{\mathrm{sym}})^{\varepsilon'_{\bar{1}} \varepsilon'_1 \tau'_{k-2} \cdots \tau'_1}_i \left(r_{\alpha_k, \ldots, \alpha_1; 2, \bar{2}} \right)^{\tau_k \tau_{k-1} \tau_{k-2} \cdots \tau_1}_{\tau'_{k-2} \cdots \tau'_1 \varepsilon_{\bar{\gamma}} \varepsilon_{\gamma}} \right\}.$$

We prove the equality by using induction on k. The cases $k = 2, 3$ is immediately follows: By using the equalities

$$(4.38) \qquad r_{\alpha_2, \alpha_1}(-1) r_{\alpha_2, 2}(-2) r_{\alpha_1, 2}(-1) = 0,$$
$$(4.39) \qquad r_{\alpha_3 \alpha_2}(-1) r_{\alpha_3 2}(-3) r_{\alpha_2 2}(-2) r_{\alpha_3 \bar{2}}(-2) r_{\alpha_2 \bar{2}}(-1) = 0,$$

we see that $\left(r_{\alpha_k, \ldots, \alpha_1; 2, \bar{2}} \right)^{\tau_k \tau_{k-1} \tau_{k-2} \cdots \tau_1}_{\tau'_{k-2} \cdots \tau'_1 \varepsilon_{\bar{\gamma}} \varepsilon_{\gamma}}$ is totally symmetric in $\tau_1, \ldots \tau_k$. For $k \geq 4$ the last expression is not totally symmetric.

The equality is clear if $i = k$ or $i = -k$ because there is only one term for the summation of $\tau_1, \ldots, \tau_k$, and t_k and t_k^{sym} are the same in this sector. Now, assume that $i \neq \pm k$.

Using the Yang-Baxter equation, and the equalities (4.38), (4.39) we see that the summand $\{\cdots\}$ in the right hand side of (4.37) is independent of $\tau_1, \tau_2, \cdots, \tau_k$. Since $i \neq \pm k$, we can choose it as $\tau_k = +$ and $\tau_{k-1} = -$. Then, the right hand side becomes

$$(4.40) \qquad \frac{4k(k-1)}{(k-i)(k+i)} \sum_{\substack{\tau_1, \ldots, \tau_{k-2} \\ \tau_1 + \cdots + \tau_{k-2} = i}} \left\{ \sum_{\tau'_1, \ldots, \tau'_{k-2}} \left(\frac{k+i}{2k} t(\lambda_2 - k)^{\varepsilon'_{\bar{1}}}_+ (t_{k-1}^{\mathrm{sym}})^{\varepsilon'_1 \tau'_{k-2} \cdots \tau'_1}_{i-1} \right. \right.$$
$$\left. \left. + \frac{k-i}{2k} t(\lambda_2 - k)^{\varepsilon'_{\bar{1}}}_- (t_{k-1}^{\mathrm{sym}})^{\varepsilon'_1 \tau'_{k-2} \cdots \tau'_1}_{i+1} \right) \left(r_{\alpha_k, \ldots, \alpha_1; 2, \bar{2}} \right)^{+ - \tau_{k-2} \cdots \tau_1}_{\tau'_{k-2} \cdots \tau'_1 \varepsilon_{\bar{\gamma}} \varepsilon_{\gamma}} \right\}.$$

Consider the first half,

$$(4.41) \qquad \frac{2(k-1)}{k-i} \sum_{\substack{\tau_1, \ldots, \tau_k \\ \tau_1 + \cdots + \tau_{k-2} = i}} \left\{ \sum_{\tau'_1, \ldots, \tau'_{k-2}} t(\lambda_2 - k)^{\varepsilon'_{\bar{1}}}_+ (t_{k-1}^{\mathrm{sym}})^{\varepsilon'_1 \tau'_{k-2} \cdots \tau'_1}_{i-1} \left(r_{\alpha_k, \ldots, \alpha_1; 2, \bar{2}} \right)^{+ - \tau_{k-2} \cdots \tau_1}_{\tau'_{k-2} \cdots \tau'_1 \varepsilon_{\bar{\gamma}} \varepsilon_{\gamma}} \right\}.$$

By the same argument as before, we see that the summand $\{\cdots\}$ is invariant for the permutation of the indices $-, \tau_{k-2}, \cdots, \tau_1$. Therefore, we can rewrite (4.41) as

$$(4.42) \qquad \sum_{\substack{\tau_1,\ldots,\tau_{k-1} \\ \tau_1+\cdots+\tau_{k-1}=i-1}} \left\{ \sum_{\tau_1',\ldots,\tau_{k-2}'} t(\lambda_2-k)_+^{\varepsilon_1'}(t_{k-1}^{\mathrm{sym}})_{i-1}^{\varepsilon_1'\tau_{k-2}'\cdots\tau_1'} \left(r_{\alpha_k,\ldots,\alpha_1;2,\bar{2}}\right)_{\tau_{k-2}'\cdots\tau_1'\varepsilon_{\bar{\gamma}}\varepsilon_{\gamma}}^{+\tau_{k-1}\tau_{k-2}\cdots\tau_1} \right\}.$$

Now, by induction hypothesis, this is equal to

$$(4.43) \qquad \sum_{\substack{\tau_1,\ldots,\tau_{k-1} \\ \tau_1+\cdots+\tau_{k-1}=i-1}} \left\{ \sum_{\tau_1',\ldots,\tau_{k-2}'} (t_k)_{+\tau_{k-1}\cdots\tau_1}^{\varepsilon_1'\varepsilon_1'\tau_{k-2}'\cdots\tau_1'} \left(r_{\alpha_k,\ldots,\alpha_1;2,\bar{2}}\right)_{\tau_{k-2}'\cdots\tau_1'\varepsilon_{\bar{\gamma}}\varepsilon_{\gamma}}^{+\tau_{k-1}\tau_{k-2}\cdots\tau_1} \right\}.$$

In order to rewrite the second half in a similar manner, we should first rewrite the indices $+-$ in $\left(r_{\alpha_k,\ldots,\alpha_1;2,\bar{2}}\right)_{\tau_{k-2}'\cdots\tau_1'\varepsilon_{\bar{\gamma}}\varepsilon_{\gamma}}^{+-\tau_{k-2}\cdots\tau_1}$ to $-+$. This is possible by the same argument again. After that, the argument is similar, and we obtain a similar expression to (4.43). Adding these two expressions, we obtain the left hand side of (4.37). $\qquad\square$

Using (4.19), we obtain

$$(4.44) \qquad X_k = d_k r_{(1\bar{1}),\bar{2}}^{(2,1)}(-k-\tfrac{1}{2})\mathrm{tr}_{\alpha_1\cdots\alpha_k}\left\{ \pi_{(\alpha_1\cdots\alpha_k)}^{(k)}\left(T^{[1,2]}(\lambda_2-\tfrac{k+1}{2})\right)\right.$$
$$\times r_{\alpha_k 2}(-k)\cdots r_{\alpha_3 2}(-3)r_{\alpha_k\bar{2}}(-k+1)\cdots r_{\alpha_3\bar{2}}(-2) \bullet$$
$$\left.\times P_{\alpha_2\bar{\gamma}}P_{\alpha_1\gamma}\mathcal{P}_{\alpha_k\cdots\alpha_1}^+\right\}r_{\bar{\gamma}2}(-2)s_{\gamma 2}^{(1)}s_{\bar{\gamma}\bar{2}}^{(1)},$$

where

$$(4.45) \qquad d_k = \frac{(-1)^k k}{(k-1)!(k+1)!\prod_{j=3}^n \lambda_{2j}(\lambda_{2j}-k-1)}.$$

We insert $\mathcal{P}_{\alpha_k\cdots\alpha_3}^+$ and $\mathcal{P}_{2\bar{2}}^+$ at the position $\bullet$ in (4.44), and using (2.17) and (2.19) obtain

$$(4.46) \qquad X_k = \frac{(k-2)!(k-1)!d_k}{2} r_{(1\bar{1}),\bar{2}}^{(2,1)}(-k-\tfrac{1}{2})$$
$$\times\mathrm{tr}_{\alpha_1\cdots\alpha_k}\left\{ \pi_{(\alpha_1\cdots\alpha_k)}^{(k)}\left(T^{[1,2]}(\lambda_2-\tfrac{k+1}{2})\right)r_{(\alpha_3\cdots\alpha_k),(2\bar{2})}^{(k-2,2)}(-\tfrac{k+2}{2})\right.$$
$$\left.\times P_{\alpha_2\bar{\gamma}}P_{\alpha_1\gamma}\mathcal{P}_{\alpha_1\cdots\alpha_k}^+\right\}r_{\bar{\gamma}2}(-2)s_{\gamma 2}^{(1)}s_{\bar{\gamma}\bar{2}}^{(1)}.$$

Now we rewrite the last part. First we use

$$(4.47) \qquad r_{\bar{\gamma}2}(-2)s_{\gamma 2}^{(1)}s_{\bar{\gamma}\bar{2}}^{(1)} = -2s_{(1\bar{1}),(2\bar{2})}^{(2)}.$$

Then, we use

$$(4.48) \qquad P_{\alpha_2\bar{\gamma}}P_{\alpha_1\gamma}\mathcal{P}_{\alpha_1,\alpha_2}^+ s_{(1\bar{1}),(2\bar{2})}^{(2)} = \frac{1}{2}r_{(\alpha_1\alpha_2),(1\bar{1})}^{(2,2)}(0)s_{(1\bar{1}),(2\bar{2})}^{(2)}$$
$$= \frac{1}{2}r_{(\alpha_1\alpha_2),(2\bar{2})}^{(2,2)}(-1)s_{(1\bar{1}),(2\bar{2})}^{(2)}.$$

where we used the crossing symmetry (2.14).

Finally, applying (2.20) to $r^{(k-2,2)}_{(\alpha_3\cdots\alpha_k),(2\bar{2})}(-\frac{k+2}{2})$ and $r^{(2,2)}_{(\alpha_1\alpha_2),(2\bar{2})}(-1)$, we obtain

(4.49)

$$
X_k = \frac{(-1)^{k+1}}{(k^2-1)\prod_{j=3}^n \lambda_{2j}(\lambda_{2j}-k-1)}
$$

$$
\times \mathrm{tr}_{\alpha_1\cdots\alpha_k}\left\{\pi^{(k)}_{(\alpha_1\cdots\alpha_k)}\left(T^{[1,2]}(\lambda_2-\frac{k+1}{2})r^{(2,1)}_{(1,\bar{1}),\bar{2}}(-k-\frac{1}{2})L^{(2)}_{(2,\bar{2})}\left(-\frac{k}{2}\right)\right)\right\}s^{(2)}_{(1,\bar{1}),(2,\bar{2})}.
$$

This expression can be further simplified using Lemma 4.1. Using the cyclicity of trace we come to the final result:

(4.50)
$$
X_k = \frac{(-1)^{k+1}}{(k+1)\prod_{j=3}^n \lambda_{2j}(\lambda_{2j}-k-1)}
$$

$$
\times \mathrm{tr}_{\alpha_1\cdots\alpha_k}\left\{\pi^{(k)}_{(\alpha_1\cdots\alpha_k)}\left(T^{[1]}(\lambda_2-\frac{k+1}{2})\right)\right\}s^{(2)}_{(1,\bar{1}),(2,\bar{2})}.
$$

Let us summarize the result by using the function ω and $X^{[1,2]}$.

Lemma 4.9. *The following difference equation holds for $X^{[1,2]}$:*

(4.51) $X^{[1,2]}(\lambda_1-1,\lambda_2,\cdots,\lambda_n)$

$$
= -\frac{(\lambda_{12}-1)(\lambda_{12}+1)}{\lambda_{12}(\lambda_{12}-2)}A_{\bar{1}}(\lambda_1,\cdots,\lambda_n)P_{1\bar{1}}X^{[1,2]}(\lambda_1,\lambda_2,\cdots,\lambda_n).
$$

Proof. Eq. (4.50) shows that

(4.52) $X_k = (-1)^k k(k+2)X^{[1,2]}(\lambda_2-k-1,\lambda_2,\ldots,\lambda_n)$

for all integers $k\geq 2$. On the other hand, from the definition of X_k we have

(4.53) $X_k = A_{\bar{1}}(\lambda_2-k,\lambda_2,\ldots,\lambda_n)P_{1\bar{1}}X_{k-1}.$

Therefore (4.51) is valid if $\lambda_{1,2}=-k$ for all integers $k\geq 2$. Since both sides of (4.51) are rational, it is valid identically. $\square$

Proposition 4.10. *For any positive integer $k\geq 1$ we have*

(4.54) $\mathrm{res}_{\lambda_1=\lambda_2-k-1}h_n(\lambda_1,\cdots,\lambda_n)_{1,\cdots,n,\bar{n},\cdots,\bar{1}}$

$$
= \mathrm{res}_{\lambda_1=\lambda_2-k-1}\{\omega(\lambda_{1,2})X^{[1,2]}(\lambda_1,\cdots,\lambda_n)\}h_{n-2}(\lambda_3,\cdots,\lambda_n)_{3,\cdots,n,\bar{n},\cdots,\bar{3}}.
$$

Proof. Note that $X^{[1,2]}(\lambda_1,\lambda_2,\ldots,\lambda_n)$ has no pole at $\lambda_1=\lambda_2-k-1$ with $k\geq 1$. Since (4.53) uniquely determines X_{k-1} from X_k for $k\geq 2$, the difference equation (4.51) implies that (4.52) is valid also for $k=1$. Eq. (4.54) is then an immediate consequence of (4.52) and

(4.55) $\mathrm{res}_{\lambda=-k-1}\omega(\lambda)=(-1)^k k(k+2).$

$\square$

It remains to find residues at poles $\lambda_1=\lambda_2+k+1$ for $k\geq 1$. The result is formulated in the

Proposition 4.11. *For any positive integer $k \geq 1$ we have*

(4.56)
$$\mathrm{res}_{\lambda_1 = \lambda_2 + k + 1} h_n(\lambda_1, \cdots, \lambda_n)_{1, \cdots, n, \bar{n}, \cdots, \bar{1}}$$
$$= \mathrm{res}_{\lambda_1 = \lambda_2 + k + 1} \{\omega(\lambda_{1,2}) X^{[1,2]}(\lambda_1, \cdots, \lambda_n)\} h_{n-2}(\lambda_3, \cdots, \lambda_n)_{3, \cdots, n, \bar{n}, \cdots, \bar{3}}.$$

Proof. Noting that $\omega(\lambda)$ is even, we have

$$\mathrm{res}_{\lambda_1 = \lambda_2 + k + 1} h_n(\lambda_1, \lambda_2 \cdots, \lambda_n)_{1, 2 \cdots n \bar{n} \cdots \bar{2}, \bar{1}}$$
$$= \mathrm{res}_{\lambda_1 = \lambda_2 + k + 1} R_{2,1}(\lambda_{21}) R_{\bar{1}, \bar{2}}(\lambda_{12}) h_n(\lambda_2, \lambda_1, \cdots, \lambda_n)_{2, 1 \cdots n \bar{n} \cdots \bar{1}, \bar{2}}$$
$$= R_{2,1}(-k-1) R_{\bar{1}, \bar{2}}(k+1) \mathrm{res}_{\lambda_1 = \lambda_2 + k + 1} \omega(\lambda_{1,2})$$
$$\times X^{[1,2]}(\lambda_2, \lambda_2 + k + 1, \cdots, \lambda_n)_{2, 1 \cdots n \bar{n} \cdots \bar{1}, \bar{2}} h_{n-2}(\lambda_3, \cdots, \lambda_n)_{3, \cdots, n, \bar{n}, \cdots, \bar{3}}$$

Hence the assertion follows from the symmetry (4.8). $\qquad \square$

Corollary 4.12. *For any $k \in \mathbb{Z} \backslash \{0, \pm 1\}$ and $2 \leq j \leq n$ we have*

(4.57)
$$\mathrm{res}_{\lambda_1 = \lambda_j - k} h_n(\lambda_1, \cdots, \lambda_n)_{1, \cdots, n, \bar{n}, \cdots, \bar{1}}$$
$$= \mathrm{res}_{\lambda_1 = \lambda_j - k} \{\omega(\lambda_{1,j}) X^{[1,j]}(\lambda_1, \cdots, \lambda_n)\} h_{n-2}(\lambda_2, \cdots, \widehat{\lambda_j}, \cdots, \lambda_n)_{2, \cdots, \hat{j}, \cdots, n, \bar{n}, \cdots, \hat{\bar{j}}, \cdots, \bar{2}}.$$

Proof. This is an immediate consequence of Proposition 4.10–4.11, the symmetry (2.29) and the relation (4.14). $\qquad \square$

4.4. **Asymptotics.** In this subsection we finish the proof of Theorem 3.1. Let $\Phi_L(\lambda_1) = h_n(\lambda_1, \cdots, \lambda_n)$, $\Phi_R(\lambda_1)$ the right hand side of (3.22), and set $\Phi(\lambda_1) = \Phi_L(\lambda_1) - \Phi_R(\lambda_1)$. We are going to show that $\Phi(\lambda_1) = 0$.

By Corollary 4.12, we have

$$\mathrm{res}_{\lambda_1 = \lambda_j - k} \Phi_L(\lambda_1) = \mathrm{res}_{\lambda_1 = \lambda_j - k} \Phi_R(\lambda_1)$$

for all $k \in \mathbb{Z} \backslash \{0, \pm 1\}$ and $j = 2, \cdots, n$. By (2.33) and the definition of $Z^{[1,j]}(\lambda_1, \cdots, \lambda_n)$, there are no other poles in λ_1. Hence $\Phi(\lambda_1)$ is an entire function.

Consider the asymptotic behavior as $\lambda_1 \to \infty$. By (2.34) we know that

$$\lim_{\substack{\lambda_1 \to \infty \\ \lambda_1 \in S_\delta}} \Phi_L(\lambda_1) = (-1)^{n-1} \frac{1}{2} s_{1, \bar{1}} h_{n-1}(\lambda_2, \cdots, \lambda_n)$$

for any $0 < \delta < \pi$, where $S_\delta = \{\lambda \in \mathbb{C} \mid \delta < |\arg \lambda_1| < \pi - \delta\}$. On the other hand, the function $\omega(\lambda)$ satisfies

$$\lim_{\substack{\lambda_1 \to \infty \\ \lambda_1 \in S_\delta}} \omega(\lambda) = 0.$$

Since the coefficients of $\omega(\lambda_{1,j})$ are regular at $\lambda_1 = \infty$ (Lemma 4.5), the terms in $\Phi_R(\lambda_1)$ except the last vanish in this limit. The last term of $\Phi_R(\lambda_1)$ is so chosen that

(4.58)
$$\lim_{\substack{\lambda_1 \to \infty \\ \lambda_1 \in S_\delta}} \Phi(\lambda_1) = 0.$$

Let us show that the condition $\lambda_1 \in S_\delta$ can be removed.

Lemma 4.13. *There exist constants $M, c > 0$ such that*

$$|\Phi(\lambda_1)| \leq M\, e^{c|\lambda_1|} \qquad (\lambda_1 \in \mathbb{C}).$$

Proof. Recall that $\Phi_L(\lambda_1)$ satisfies the difference equation of the form $\Phi_L(\lambda_1 + 1) = B(\lambda_1)\Phi_L(\lambda_1)$, where $B(\lambda_1)$ is a matrix of rational functions holomorphic and invertible at $\lambda_1 = \infty$. Take a small neighborhood U of the set of poles of $B(\lambda_1)^{\pm 1}$, and choose $K' > 0$ such that $\sup_{\lambda_1 \in \mathbb{P}^1 \setminus U} |B(\lambda_1)^{\pm 1}| \leq K'$. Choose $\lambda_1^0 \in \mathbb{C}$ so that $\lambda_1^0 + \mathbb{Z} + i\mathbb{R} \subset \mathbb{P}^1 \setminus U$. Then for $n \geq 0$ we have

$$(4.59) \quad \begin{aligned} |\Phi_L(\lambda_1^0 + it + n)| &\leq |B(\lambda_1^0 + it + n - 1)| \cdots |B(\lambda_1^0 + it)||\Phi_L(\lambda_1^0 + it)| \\ &\leq M'K'^n, \end{aligned}$$

where $M' = \sup_{t \in \mathbb{R}} |\Phi_L(\lambda_1^0 + it)|$. Clearly a similar estimate holds also for $|\Phi_L(\lambda_1^0 + it - n)|$.

On the other hand, the function $\omega(\lambda)$ satisfies the difference equation

$$\begin{pmatrix} \omega(\lambda + 1) \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{\lambda(\lambda+2)}{\lambda^2-1} & \frac{3}{2(\lambda^2-1)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \omega(\lambda) \\ 1 \end{pmatrix}.$$

Hence by the same argument as above, we obtain an estimate for $\omega(\lambda_{1,j})$, and hence for $\Phi_R(\lambda_1)$, of the type (4.59).

In summary, there exist $M, K > 0$ such that

$$\sup_{t \in \mathbb{R}} |\Phi(\lambda_1^0 + it + n)| \leq MK^{|n|} \qquad (n \in \mathbb{Z}).$$

The Lemma follows from this and the maximum principle. $\qquad\square$

Now fix $\delta < \pi/2$. By (4.58) there exists an $M' \geq M$ such that

$$(4.60) \qquad |\Phi(\lambda_1)| \leq M' \qquad (\lambda_1 \in \mathbb{C}, \delta \leq |\arg \lambda_1| \leq \pi - \delta).$$

By the lemma above and the Phragmen-Lindelöf theorem, (4.60) holds also for $|\arg \lambda_1| \leq \delta$ or $\pi - \delta \leq |\arg \lambda_1| \leq \pi$. Therefore $\Phi(\lambda_1)$ is bounded in the full neighborhood of $\lambda_1 = \infty$. Hence $\lambda_1 = \infty$ is a regular point of $\Phi(\lambda_1)$, and we conclude that $\Phi(\lambda_1) = 0$. This completes the proof of Theorem 3.1.

Appendix A. Relation to the correlators of the XXZ model

We give the relation between two different gauges, the one used in the XXZ model in the massive regime [12], and the one used in the present paper, which gives the Hamiltonian (2.1).

In [12], the following Hamiltonian was considered:

$$H_{XXZ} = -\frac{1}{2} \sum_{j=1}^{L} \left(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y - \frac{x + x^{-1}}{2} \sigma_j^z \sigma_{j+1}^z \right).$$

Here $q = -x$, the XXX limit is $x \to 1$. The gauge transformation by $\mathcal{K} = \prod_{j:\text{even}} \sigma_j^z$ brings H_{XXZ} to

$$
\begin{aligned}
\widetilde{H}_{XXZ} &= \mathcal{K} H_{XXZ} \mathcal{K}^{-1} \\
&= \frac{1}{2} \sum_{j=1}^{L} \left(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \frac{x + x^{-1}}{2} \sigma_j^z \sigma_{j+1}^z \right).
\end{aligned}
$$

The ground states and the correlators are related as

$$
|\widetilde{\Omega}\rangle = \mathcal{K}|\Omega\rangle,
$$
$$
\langle \widetilde{\Omega}|(E_{\epsilon_1,\bar{\epsilon}_1})_1 \cdots (E_{\epsilon_n,\bar{\epsilon}_n})_n|\widetilde{\Omega}\rangle = \prod_{\substack{1 \le j \le n \\ j:\text{even}}} \epsilon_j \bar{\epsilon}_j \cdot \langle \Omega|(E_{\epsilon_1,\bar{\epsilon}_1})_1 \cdots (E_{\epsilon_n,\bar{\epsilon}_n})_n|\Omega\rangle.
$$

$$
G_{2n}(\zeta_1, \cdots, \zeta_n, x\zeta_n, \cdots, x\zeta_1)^{-\epsilon_1, \cdots, -\epsilon_n, \bar{\epsilon}_n, \cdots, \bar{\epsilon}_1} = \langle \Omega|(E_{\epsilon_1,\bar{\epsilon}_1})_1 \cdots (E_{\epsilon_n,\bar{\epsilon}_n})_n|\Omega\rangle.
$$

$$
g_{2n}(\zeta_1, \cdots, \zeta_{2n}) = (-1)^n \prod_{\substack{1 \le j \le 2n \\ j:\text{even}}} \sigma_j^z \cdot G_{2n}(\zeta_1, \cdots, \zeta_{2n}).
$$

Therefore,

$$
\begin{aligned}
&g_{2n}(\zeta_1, \cdots, \zeta_n, x\zeta_n, \cdots, x\zeta_1)^{-\epsilon_1, \cdots, -\epsilon_n, \bar{\epsilon}_n, \cdots, \bar{\epsilon}_1} \\
&= (-1)^{[n/2]} \prod_{j=1}^{n}(-\bar{\epsilon}_j) \cdot \langle \widetilde{\Omega}|(E_{\epsilon_1,\bar{\epsilon}_1})_1 \cdots (E_{\epsilon_n,\bar{\epsilon}_n})_n|\widetilde{\Omega}\rangle.
\end{aligned}
$$

The Hamiltonian $\widetilde{H}_{XXZ}$ and the ground state $|\widetilde{\Omega}\rangle$ are related to those in Section 2 by

$$
\lim_{x \to 1} \widetilde{H}_{XXZ} = H_{XXX}, \quad \lim_{x \to 1} |\widetilde{\Omega}\rangle = |\text{vac}\rangle.
$$

Therefore, we have the relation (2.2).

Appendix B. Analyticity and asymptotic properties of h_n

Here we give a proof for Proposition 2.2. We start from the integral formula for the function $h_n(\lambda_1, \ldots, \lambda_n)^{\epsilon_1, \ldots, \epsilon_n, \bar{\epsilon}_n, \ldots, \bar{\epsilon}_1}$ given in [12].

Set

$$
A = \{j | \epsilon_j = +\}, \quad B = \{j | \bar{\epsilon}_j = +\}.
$$

We write $A = \{a_1, \ldots, a_r\}, B = \{b_1, \ldots, b_s\}$ with $a_1 < \cdots < a_r, b_1 < \cdots < b_s$. Note that $r + s = n$. For each $a \in A$ and $b \in B$ we prepare an integration variable t_a and t_b', respectively. Arrange the variables as

$$
(u_1, \ldots, u_n) = (t_{a_r}, \ldots, t_{a_1}, t_{b_1}', \ldots, t_{b_s}').
$$

Then the formula is given by

(B.1) $h_n(\lambda_1, \ldots, \lambda_n)^{\epsilon_1, \ldots, \epsilon_n, \bar{\epsilon}_n, \ldots, \bar{\epsilon}_1}$

$$
= c_{A,B}^{(n)} \prod_{a \in A} \int_{C^+} \frac{dt_a}{t_a - \lambda_a} \prod_{b \in B} \int_{C^-} \frac{dt_b'}{t_b' - \lambda_b}
$$

$$
\times \prod_{\substack{a \in A \\ j < a}} \frac{t_a - \lambda_j - 1}{t_a - \lambda_j} \prod_{\substack{b \in B \\ j < b}} \frac{t_b' - \lambda_j + 1}{t_b' - \lambda_j}
$$

$$
\times \prod_{j < k} \left(\frac{\sinh \pi i(u_j - u_k)}{u_j - u_k - 1} \frac{\sinh \pi i(\lambda_j - \lambda_k)}{\lambda_j - \lambda_k} \right) \prod_{j,k} \frac{u_j - \lambda_k}{\sinh \pi i(u_j - \lambda_k)}.
$$

Here $c_{A,B}^{(n)}$ is a constant which depends on A, B and n. The integration contour C^+ is parallel to the imaginary axis for $|\mathrm{Im} t_a| \gg 0$, and separates the sequences of the poles of the integrand into the two sets $\lambda_j + \mathbb{Z}_{\leq 0}$ and $\lambda_j + \mathbb{Z}_{>0}$. Similarly C^- is the contour separating the poles into $\lambda_j + \mathbb{Z}_{<0}$ and $\lambda_j + \mathbb{Z}_{\geq 0}$.

Now let us check the analyticity property (2.33). The singularity of the integral (B.1) comes from the pinch of the integration contour by some poles of the integrand. Hence the function h_n is meromorphic with at most poles at $\lambda_i - \lambda_j \in \mathbb{Z} \setminus \{0\}$. Let us prove that h_n is analytic at $\lambda_i = \lambda_j \pm 1$. From (2.29) it suffices to prove that h_n is analytic at $\lambda_1 = \lambda_2 \pm 1$. The function h_n satisfies (2.30), and $A_{\bar{1}}(\lambda_1, \lambda_2, \ldots)$ and $h_n(\lambda_1, \lambda_2, \ldots)$ are regular at $\lambda_1 = \lambda_2$. Hence h_n is analytic at $\lambda_1 = \lambda_2 - 1$. Similarly the regularity at $\lambda_1 = \lambda_2 + 1$ can be checked by using the equation

$$
h_n(\lambda_1 + 1, \lambda_2, \ldots, \lambda_n)_{1, \ldots, n, \bar{n}, \ldots, \bar{1}}
$$

$$
= \frac{-1}{\prod_{j=2}^n (\lambda_{1j}^2 - 1)} r_{12}(-\lambda_{12} - 1) \cdots r_{1n}(-\lambda_{1n} - 1) r_{1\bar{n}}(-\lambda_{1n}) \cdots r_{1\bar{2}}(-\lambda_{12})
$$

$$
\times h_n(\lambda_1, \ldots, \lambda_n)_{\bar{1}, 2, \ldots, n, \bar{n}, \ldots, \bar{2}, 1},
$$

which is derived from (2.30). It is easy to check that the poles at $\lambda_i - \lambda_j \in \mathbb{Z} \setminus \{0, \pm 1\}$ are simple by using (2.30) repeatedly.

Next we show the asymptotic property (2.34). We consider the asymptotics in the angular domain

(B.2) $\lambda_1 \to \infty, \quad \delta < |\arg \lambda_1| < \pi - \delta,$

where $0 < \delta < \pi$. Setting $\lambda_1 = \kappa + i\mu$ with $\kappa, \mu \in \mathbb{R}$ we have in this domain

$$
|\mu| \leq |\lambda_1| \leq K|\mu|
$$

for a positive constant K. We will consider only the case $\mu \to +\infty$ since the other case is similar.

First we deform the contours $C^\pm$ suitably and assume that the contours $C^\pm, C^\pm + 1$ and $C^\pm - 1$ do not cross each other. Taking $\mu \gg 0$ we also assume that $|\mathrm{Im} \lambda_j| \leq \mu/6$ for $j = 2, \ldots, n$, and that $C^\pm$ contains the segment

$$
\Gamma^\pm = \{\kappa \pm \epsilon + iy | \frac{2}{3}\mu \leq y \leq \frac{4}{3}\mu\}
$$

for small $\epsilon > 0$.

Denote by $P_{A,B}$ the rational part of the integrand, that is

$$P_{A,B}(u_1, \ldots, u_n; \lambda_1, \ldots, \lambda_n)$$

$$= \prod_{a \in A} \left(\frac{1}{t_a - \lambda_a} \prod_{j<a} \frac{t_a - \lambda_j - 1}{t_a - \lambda_j} \right) \prod_{b \in B} \left(\frac{1}{t'_b - \lambda_b} \prod_{j<b} \frac{t'_b - \lambda_j + 1}{t'_b - \lambda_j} \right)$$

$$\times \frac{\prod_{j,k}(u_j - \lambda_k)}{\prod_{j<k}(u_j - u_k - 1)(\lambda_j - \lambda_k)}.$$

We use

$$\prod_{j<k} \sinh \pi i (u_j - u_k) = 2^{-\binom{n-2}{2}} e^{\pi i (n-1)\lambda_1}$$

$$\times \quad \text{Skew} \left(\prod_{j=1}^{n-1} e^{\pi i (n-2j)u_j} \sinh \pi i (u_j - \lambda_1) \cdot e^{\pi i (-n+1)u_n} \right),$$

where Skew is the skew-symmetrization with respect to $u_1, \ldots, u_n$. Then we find

(B.3)
$$h_n(\lambda_1, \ldots, \lambda_n)^{\epsilon_1, \ldots, \epsilon_n, \bar{\epsilon}_n, \ldots, \bar{\epsilon}_1}$$

$$= 2^{-\binom{n-2}{2}} c_{A,B}^{(n)} e^{\pi i (n-1)\lambda_1} \prod_{j<k} \sinh \pi i (\lambda_j - \lambda_k)$$

$$\times \sum_{\sigma \in \mathfrak{S}_n} (\text{sgn}\sigma) \prod_{j=1}^{n} \int_{C_{\sigma,j}} du_j \; P_{A,B}(u_{\sigma(1)}, \ldots, u_{\sigma(n)}; \lambda_1, \ldots, \lambda_n)$$

$$\times \prod_{j=1}^{n-1} \left(\frac{e^{\pi i (n-2j)u_j}}{\prod_{k=2}^{n} \sinh \pi i (u_j - \lambda_k)} \right) \frac{e^{\pi i (-n+1)u_n}}{\prod_{k=1}^{n} \sinh \pi i (u_n - \lambda_k)}.$$

Here the contour $C_{\sigma,j}$ is determined by

$$C_{\sigma,j} = \begin{cases} C^+ & \text{if } 1 \le \sigma^{-1}(j) \le r, \\ C^- & \text{if } r+1 \le \sigma^{-1}(j) \le n. \end{cases}$$

Note that the factor $e^{\pi i (n-1)\lambda_1} \prod_{j<k} \sinh \pi i (\lambda_j - \lambda_k)$ converges when $\mu \to +\infty$. In the following we prove that each integral in the sum of the right hand side of (B.3) converges in the limit as $\mu \to +\infty$.

Decompose the contour for u_n into two parts $C_{\sigma,n} = \Gamma + C'$, where Γ is the segment Γ^+ or Γ^- contained in $C_{\sigma,n}$, and $C' = C_{\sigma,n} \setminus \Gamma$. Simultaneously we decompose the integral

$$\prod_{j=1}^{n} \int_{C_{\sigma,j}} du_j = \prod_{j=1}^{n-1} \int_{C_{\sigma,j}} du_j \int_{\Gamma} du_n + \prod_{j=1}^{n-1} \int_{C_{\sigma,j}} du_j \int_{C'} du_n.$$

Further we decompose the integration in the first term over $C_{\sigma,1} \times \cdots \times C_{\sigma,n-1}$ into the following parts:

$$D_1 = \{(u_1, \ldots, u_{n-1}) \in C_{\sigma,1} \times \cdots \times C_{\sigma,n-1} | -\mu/3 \le \text{Im} u_j \le \mu/3\},$$
$$D_2 = (C_{\sigma,1} \times \cdots \times C_{\sigma,n-1}) \setminus D_1.$$

Thus the integral in (B.3) is decomposed as

$$
(B.4) \qquad \int_{D_1} \prod_{j=1}^{n-1} du_j \int_{\Gamma} du_n + \int_{D_2} \prod_{j=1}^{n-1} du_j \int_{\Gamma} du_n + \prod_{j=1}^{n-1} \int_{C_{\sigma,j}} du_j \int_{C'} du_n.
$$

We consider the limit of these three parts separately.

Let us consider the first integral of (B.4). Change the variable $u_n \mapsto u_n + \lambda_1$. Then the integral is equal to

$$
(B.5) \qquad \int_{D_1} \prod_{j=1}^{n-1} du_j \int_{\pm\epsilon-\frac{\mu}{3}i}^{\pm\epsilon+\frac{\mu}{3}i} du_n P_{A,B}(\ldots, u_n + \lambda_1, \ldots; \lambda_1, \ldots, \lambda_n)
$$

$$
\times \prod_{j=1}^{n-1} \left(\frac{e^{\pi i(n-2j)u_j}}{\prod_{k=2}^{n} \sinh \pi i(u_j - \lambda_k)} \right) \frac{1}{\sinh \pi i u_n} \prod_{k=2}^{n} \frac{e^{-\pi i(u_n+\lambda_1)}}{\sinh \pi i(u_n + \lambda_1 - \lambda_k)},
$$

where the sign $\pm\epsilon$ is $+$ or $-$ according to whether $\Gamma=\Gamma^+$ or Γ^-, respectively.

Note that

$$
P_{A,B}(\ldots, u_n + \lambda_1, \ldots; \lambda_1, \ldots, \lambda_n) = O(1), \quad (\lambda_1 \to \infty).
$$

Hence the integrand converges in the limit $\mu \to +\infty$ for fixed $u_1, \ldots, u_n$. To apply Lebesgue's convergence theorem, let us check that the integrand is bounded from above by an integrable function. First we consider the rational part $P_{A,B}$. Recall that the contours $C^{\pm}, C^{\pm}-1$ and $C^{\pm}+1$ do not cross each other. Hence there exists a positive constant d such that

$$
(B.6) \qquad |u_j - u_k \pm 1| \geq d
$$

for $j, k = 1, \ldots, n-1$. For $(u_1, \ldots, u_{n-1}) \in D_1$ and $u_n = \pm\epsilon+iy, (-\mu/3 \leq y \leq \mu/3)$, we have

$$
|u_j - u_n - \lambda_1 \pm 1| \geq |\text{Im}(u_j - u_n - \lambda_1)| \geq \mu/3 \quad (j = 1, \ldots, n-1).
$$

Hence the rational part $P_{A,B}$ is upper bounded as

$$
|P_{A,B}(\ldots, u_n + \lambda_1, \ldots; \lambda_1, \ldots, \lambda_n)|
$$
$$
\leq \mu^{-2(n-1)} Q_1(|u_1|, \ldots, |u_n|; |\lambda_2|, \ldots, |\lambda_n|; K; \mu),
$$

where Q_1 is a polynomial such that $\deg_\mu Q_1 = 2(n - 1)$. Next we consider the trigonometric part. Note that the function $e^{\pi i x}/\sinh \pi i x$ is bounded in $\mathbb{C} \setminus U$, where U is a union of a small open disk with a fixed radius around integer points. Hence we have

$$
(B.7) \qquad \left| \prod_{j=1}^{n-1} \left(\frac{e^{\pi i(n-2j)u_j}}{\prod_{k=2}^{n} \sinh \pi i(u_j - \lambda_k)} \right) \frac{1}{\sinh \pi i u_n} \prod_{k=2}^{n} \frac{e^{-\pi i(u_n+\lambda_1)}}{\sinh \pi i(u_n + \lambda_1 - \lambda_k)} \right|
$$

$$
\leq M \left| \prod_{j=1}^{n-1} \frac{1}{\sinh \pi i(u_j - \lambda_2)} \frac{1}{\sinh \pi i u_n} \right|
$$

for some positive constant M. The right hand side decreases exponentially as $\text{Im} u_j \to \pm\infty$. Therefore we can apply Lebesgue's convergence theorem to (B.5).

Next consider the second integral in (B.4). Change the variable $u_n \mapsto u_n + \lambda_1$. Then the integral is equal to (B.5) with D_1 replaced by D_2. Let us prove that the integral vanishes in the limit $\mu \to +\infty$. To prove this we decompose D_2 as follows. Set

$$U_\pm^{(k)} = \{\pm\mathrm{Im}\,u_j > \mu/3\} \cap D_2$$

for $k = 1, \ldots, n-1$. Then $D_2 = \cup_k (U_+^{(k)} \cup U_-^{(k)})$. We prove that the integral over each set $U_\pm^{(k)} \times \{\pm\epsilon + iy| -\mu/3 \le y \le \mu/3\}$ vanishes in the limit. Here we consider the case of $k = 1$. The argument is similar for the other cases. To prove the vanishing, we construct an upper bound for the integrand by a certain integrable function of the form

$$\text{(B.8)} \qquad e^{-c\mu} \sum_{k=-(n-1)}^{n-1} \mu^k R_k(u_1, \ldots, u_n; \lambda_2, \ldots, \lambda_n),$$

where c is a positive constant. First consider the rational part $P_{A,B}$. We have the inequality (B.6) and

$$|u_j - u_n - \lambda_1 \pm 1| \ge d$$

for $j = 1, \ldots, n - 1$. Hence we have an upper estimate of the form

$$|P_{A,B}(\ldots, u_n + \lambda_1, \ldots; \lambda_1, \ldots, \lambda_n)|$$
$$\le \mu^{-(n-1)} Q_2(|u_1|, \ldots, |u_n|; |\lambda_2|, \ldots, |\lambda_n|; K; \mu),$$

where Q_2 is a polynomial such that $\deg_\mu Q_2 = 2(n-1)$. Next consider the trigonometric part. We have the inequality (B.7). Apply the following inequality to the factor $1/\sinh \pi i(u_1 - \lambda_2)$:

$$\left| \frac{1}{\sinh \pi i u} \right| \le \frac{2e^{-\pi|\mathrm{Im}\,u|}}{1 - e^{-\pi\mu/3}} \le \frac{2e^{-\pi\mu/12} \cdot e^{-\pi|\mathrm{Im}\,u|/3}}{1 - e^{-\pi\mu/3}} \qquad (\pm\mathrm{Im}\,u > \mu/6 > 0).$$

Thus we get the upper bound of the form (B.8).

Finally let us consider the third integral in (B.4). It also vanishes in the limit $\mu \to +\infty$ as follows. The integrand is given in (B.3). By the same argument above we have an upper bound for the rational part $P_{A,B}$ of the form

$$\mu^{-(n-1)} Q_3(|u_1|, \ldots, |u_n|; |\lambda_2|, \ldots, |\lambda_n|; K; \mu),$$

where Q_3 is a polynomial such that $\deg_\mu Q_3 = n - 1$. The trigonometric part is estimated as follows:

$$\left| \prod_{j=1}^{n-1} \left(\frac{e^{\pi i(n-2j)u_j}}{\prod_{k=2}^{n} \sinh \pi i(u_j - \lambda_k)} \right) \frac{e^{\pi i(-n+1)u_n}}{\prod_{k=1}^{n} \sinh \pi i(u_n - \lambda_k)} \right|$$
$$\le M \left| \prod_{j=1}^{n-1} \frac{1}{\sinh \pi i(u_j - \lambda_2)} \frac{1}{\sinh \pi i(u_n - \lambda_1)} \right|.$$

Now apply (B.9) to the factor $1/\sinh \pi i(u_n - \lambda_1)$, and we can see the vanishing in the limit $\mu \to +\infty$.

From the consideration above we find that h_n converges in the limit (B.2). It remains to show the limit is equal to the right hand side of (2.34). Denote the limit by $\hat{h}_n(\lambda_2, \ldots, \lambda_n)$. From the equation (2.30), using

$$\lim_{\lambda_1 \to \infty} A_{\bar{1}}(\lambda_1, \ldots, \lambda_n) = -1,$$

we obtain

$$\hat{h}_n(\lambda_2, \ldots, \lambda_n)_{1,2,\cdots,n,\bar{n},\cdots,\bar{2},\bar{1}} = -\hat{h}_n(\lambda_2, \ldots, \lambda_n)_{\bar{1},2,\cdots,n,\bar{n},\cdots,\bar{2},1}.$$

Namely, the limit $\hat{h}_n$ is a singlet in the space $V_1 \otimes V_{\bar{1}}$. From this and (2.32), we get (2.34).

References

[1] R. Baxter, Exactly solved models in statistical mechanics *Academic Press* (1982).

[2] V. Bazhanov, S. Lukyanov and A. Zamolodchikov, Integrable structure of conformal field theory II.*Q*-operator and DDV equation, *Commun.Math.Phys.***190** (1997) 247–278.

[3] H. Bethe, Zur Theorie der Metalle. I. Eigenwerte und Eigenfunktionen der linearen Atomkette, Zeitschrift für Physik **71** (1931) 205.

[4] H. Boos and V. Korepin, Quantum spin chains and Riemann zeta functions with odd arguments, hep-th/0104008, *J. Phys.* **A 34** (2001) 5311–5316.

[5] H. Boos, V. Korepin and F. Smirnov, Emptiness formation probability and quantum Knizhnik-Zamlodchikov equation, hep-th/0209246, *Nucl. Phys. B* Vol. 658/3 (2003) 417 –439.

[6] H. Boos, V. Korepin and F. Smirnov, New formulae for solutions of quantum Knizhnik-Zamolodchikov equations on level −4, hep-th/0304077, *J. Phys.* **A 37** (2004) 323–336.

[7] H. Boos, V. Korepin and F. Smirnov, New formulae for solutions of quantum Knizhnik-Zamolodchikov equations on level −4 and correlation functions, hep-th/0305135,

[8] H. Boos, M. Shiroishi and M. Takahashi, "First principle approach to correlation functions of spin-1/2 Heisenberg chain: fourth-neighbor correlators", work in progress.

[9] L. Faddeev, How algebraic Bethe Ansatz works for integrable models, *Les Houches 1995, Relativistic gravitation and gravitational radiation*, 149–219.

[10] L. Faddeev and L. Takhtajan Spectrum and scattering of excitations in the one-dimensional isotropic Heisenberg model, *J.Sov.Math.***24**(1984), 241–267.

[11] I. Frenkel and N. Reshetikhin, Quantum affine algebras and holonomic difference equations, *Commun. Math. Phys.* **146** (1992), 1–60.

[12] M. Jimbo and T. Miwa, *Algebraic Analysis of Solvable Lattice Models*, CBMS Regional Conference Series in Mathematics vol.**85**, AMS, 1994.

[13] M. Jimbo, T. Miwa, K. Miki and A. Nakayashiki, Correlation functions of the XXZ model for $\Delta < -1$, *Phys. Lett. A* **168** (1992), 256–263.

[14] G. Kato, M. Shiroishi, M. Takahashi, K. Sakai, Third-neighbor and other four-point correlation functions of spin-1/2 XXZ chain, *J. Phys. A: Math. Gen.* **36** (2003) L337.

[15] G. Kato, M. Shiroishi, M. Takahashi, K. Sakai, Next Nearest-Neighbor Correlation Functions of the Spin-1/2 XXZ Chain at Critical Region, *J. Phys. A: Math. Gen.* **37** (2004) 5097.

[16] N. Kitanine, J.-M. Maillet and V. Terras, Correlation functions of the XXZ Heisenberg spin-$\frac{1}{2}$-chain in a magnetic field, *Nucl. Phys. B* **567** (2000), 554–582.

[17] V. Korepin, A. Izergin, F. Essler and D. Uglov, Correlation functions of the spin-1/2 XXX antiferromagnet, *Phys. Lett.* **A190** (1994), 182–184.

[18] K. Sakai, M. Shiroishi, Y. Nishiyama, M. Takahashi, Third neighbor correlators of a one-dimensional spin-1/2 Heisenberg antiferromagnet, *Phys. Rev.* E **67** (2003) 065101.

[19] F. Smirnov, Form factors in completely integrable models in quantum field theory, World Scientific, Singapore, 1992.

[20] F.A. Smirnov, Dynamical symmetries of massive integrable models, Int. J. Mod. Phys. A7 (1992) S813-858

[21] M. Takahashi, Half-filled Hubbard model at low temperature, *J. Phys. C* **10**(1977) 1298

[22] M. Takahashi, G. Kato, M. Shiroishi, Next Nearest-Neighbor Correlation Functions of the Spin-1/2 XXZ Chain at Massive Region, *J. Phys. Soc. Jpn.* **73** (2004) 245.

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